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VISCOELASTIC CONSTITUTIVE LAWS FOR ICE

Final Technical Report

by

L. W. Morland ~~and~~ U. Spring

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Frame indifferent tensor relations between stress, stress-rate, strain-rate, and acceleration, and between stress, stress-rate, strain, and strain-rate, are constructed to describe the observed response of ice to maintained constant stress and to maintained constant strain-rate in uni-axial stress. In the former class of relation, which defines a viscoelastic fluid model, it is shown that two types of response reflect both common and independent

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properties, and that no simpler differential law can describe the observed features. In the latter class, which defines a viscoelastic solid model, and possible initial or induced anisotropy, it is again shown that no simpler differential law can describe the observed features, but that the two types of response are not sufficient to determine the model. A small strain approximation is deduced.

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SUMMARY

A frame indifferent tensor relation between stress, stress-rate, strain-rate, and strain-acceleration is constructed to describe the response of ice to maintained constant stress and to maintained constant strain-rate in uni-axial stress. It is shown that the two types of response reflect some common properties of this viscoelastic fluid law, but that each type contains further independent features, so that both are necessary to determine the required response functions. The shape of the tensor relation cannot be determined by uni-axial response. A reduced model with response functions of single argument is constructed, and correlated with an idealised family of constant stress responses. The extent to which the reduced model can reproduce the complete response is illustrated for two different families. Next a viscoelastic solid law relating stress, stress-rate, strain and strain-rate is shown to be compatible with the same uni-axial responses, but dependence on the reference configuration through strain allows a description of initial or induced anisotropy. A small strain approximation is constructed. However, this minimial form involves three independent response functions for an initially isotropic solid, and constant load and constant displacement rate responses provide only two relations. A further restriction must be imposed to complete the description, but the simplification to constant modulus at constant strain-rate is shown to be incompatible with expected response. We are now investigating alternative simplifications. Note the contrast with the minimal fluid model which is overdetermined by independent constant stress and constant strain-rate responses.

Contents

Keywords

Creep of ice

Constant stress and constant strain-rate response in uni-axial stress

Viscoelastic fluid laws of differential type

Viscoelastic solid laws of differential type

Viscoelastic Fluid Law For Ice

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Abstract

A frame-indifferent differential operator law relating stress, stress-rate, strain-rate, and strain-acceleration, is constructed to describe the qualitative features of both constant stress and constant strain-rate response in uni-axial stress experiments. The structure of the tensor relation cannot be determined by uni-axial data, and a variety of models reducing to the required uni-axial form are presented. The differential law allows exact description of a family of strain-rate curves at different constant uni-axial stress, and some features of the stress curves at different constant strain-rates, showing that constant stress and constant strain-rate response reflect some common material properties, but also reflect independent properties. That is, both types of response are necessary to define the material. A reduced model which adopts a restricted form of dependence on stress and strain-rate invariants is also analysed. Idealised families of constant-stress responses are constructed and various forms of limited matching to the reduced model are presented to show the extent to which the simplified law can describe the complete response.

Introduction

In ice engineering the mechanical response of ice under maintained stress is required over times sufficient for creep to strains of a few percent, or to rupture at appropriate stress combinations. This covers the instantaneous elastic strain, primary decelerating creep, secondary or approximately steady creep, and some part of the accelerating tertiary creep which on glaciological time scales is supposed to approach steady creep. The Young's modulus and bulk modulus of ice are of order $10^{10}Nm^{-2}$ (Sinha 1978, Mellor 1980), so that a moderate uni-axial compressive stress of $10^{6}Nm^{-2}$ (10 bars) induces an elastic strain of only 10^{-4} compared with creep strains of 10^{-2} common in application, and it is convenient to neglect the elastic response and model only the creep. The constitutive law may still be of "solid type", depending on a reference configuration, which is essential to model anisotropy. While both initial and induced anistropy can be significant there has been no systematic investigation (Mellor 1980), so a realistic constitutive model is not yet feasible. Here we focus on a differential operator "fluid type" model to describe observed viscoelastic creep, necessarily isotropic in all configurations.

Constant uni-axial stress experiments to determine the short time, small strain, creep have been reported by Sinha (1978a, 1978b) and Gold and Sinha (1980). Total creep times were typically of several minutes duration with maximum strains of order 3×10^{-4}, but a wide range of constant

temperature, 228K to 263K, was covered. The results are used to infer an empirical relation for the uni-axial strain at time t in terms of the stress and time explicitly, which incorporates instantaneous elastic response, recoverable creep, and viscous flow. It is not shown how the relation can be related to a three-dimensional tensor law (coordinate invariant) necessary for combined stress loading, nor is it clear how the response to a non-constant stress history is deduced since there is no superposition principle for non-linear response. We will now examine a frame indifferent tensor relation which contains the minimum dependence on rates necessary to describe the main qualitative features of uni-axial response. As demonstrated in an earlier more limited treatment (Morland 1979), uni-axial response cannot determine the shape of a tensor relation, which would require tri-axial (three independent principal stresses), or combined shear and axial stress, tests. There are an infinity of tensor relations which reduce to the same uni-axial response, so that construction of an explicit tensor relation requires many more assumptions about shape and the arguments of response functions. The various restrictions and possible alternatives are noted in our illustrative construction.

Mellor (1980) has described qualitatively the uni-axial response at constant temperature under constant compressive stress and under constant compression strain-rate. He notes that data from both tests must reflect the same material properties, but points out that no common rheological model simulates the observed responses in both tests. We now construct

a differential operator viscoelastic fluid law which describes the main features of the responses in both tests, showing that neither the constant stress nor constant strain-rate data can determine the law completely, so that these two forms of response do indeed contain independent information and neither reflects fully the mechanical properties. This contrasts with the theory of linear viscoelasticity for which the existence of a linear hereditary integral law (Gurtin and Sternberg, 1962) implies that a unique description is determined by the creep functions, namely response under constant stress.

Figure 1 shows the axial strain-rate $r(t)$ for a uni-axial compressive stress σ applied at time $t = 0$ and maintained constant. In the primary creep am, r decreases from the initial strain-rate $r_o(\sigma)$ to the minimum strain-rate $r_m(\sigma)$, both, in general, depending on the stress level σ, and the turning point m corresponds to the inflexion point (or secondary creep) on a strain-time curve. In the tertiary creep me, r increases from $r_m(\sigma)$, and we suppose an equilibrium strain-rate $r_e(\sigma)$ is attained asymptotically, corresponding to the commonly assumed steady creep appropriate to many glaciology applications. In Fig. 1 we have shown $r_e(\sigma) < r_o(\sigma)$, and will present analysis for this case, but a similar treatment can be given for the case $r_e(\sigma) > r_o(\sigma)$. The time to minimum strain-rate is $t_m(\sigma)$, which increases significantly as $\sigma \to 0$ so that minimum strain-rate is not attained in many low stress laboratory tests. A steady creep law for glaciology applications should be the tertiary limit

$r_e(\sigma)$, not the minimum strain-rate $r_m(\sigma)$, but Mellor (1980) suggests that the distinction diminishes as stress decreases and may not be significant at deviatoric stresses below $10^5 Nm^{-2}$ (1 bar) typical of many glaciology situations.

Figure 2 shows the uni-axial compressive stress $\sigma(t)$ when the axial strain-rate r is held constant. In the first stage OM, σ rises from zero to a maximum stress $\sigma_M(r)$ at time $t_M(r)$, then decreases in the stage ME, to an asymptotic equilibrium limit $\sigma_E(r)$ for consistency with a tertiary creep limit $r_e(\sigma)$. Mellor (1980) also suggests that $\sigma_E \to \sigma_M$ as strain-rate $\to 0$. The assumption of a steady (finite rate) creep $r_e(\sigma)$ as $t \to \infty$ at constant stress σ implies a correspondence with the limit $\sigma_E(r)$ at constant strain-rate r given by

$$\begin{aligned} & r_e[\sigma_E(r)] = r \quad \text{and} \quad \sigma_E[r_e(\sigma)] = \sigma \ , \\ \text{or} \quad & r_e^{-1}(r) = \sigma_E(r) \quad \text{and} \quad \sigma_E^{-1}(\sigma) = r_e(\sigma) . \end{aligned} \tag{1}$$

Mellor (1980) suggests that there are indications that the maximum stress at a given constant strain-rate is the constant stress required to produce that strain-rate as the minimum. We will adopt this property, thus

$$\begin{aligned} & r_m[\sigma_M(r)] = r \quad \text{and} \quad \sigma_M[r_m(\sigma)] = \sigma \ , \\ \text{or} \quad & r_m^{-1}(r) = \sigma_M(r) \quad \text{and} \quad \sigma_M^{-1}(\sigma) = r_m(\sigma) \ . \end{aligned} \tag{2}$$

Both inverse relations (1) and (2) are illustrated in Fig. 3. Mellor further speculates that the strain $\varepsilon_m(\sigma)$ at the minimum strain-rate in the constant stress test, and the strain $\varepsilon_M(r)$ at the maximum stress in the constant strain-rate test,

do not vary significantly with σ and r respectively over some range of σ and r, and are approximately the same, slightly less than 0·01.

While corresponding tensile tests are not practical over a wide stress range it is expected that tensile response will differ in magnitude, and be limited by tensile fracture. In our simple illustrative model we restrict correlation to the compression responses shown by Figs 1 and 2, but indicate how distinct independent compression and tension response (if known) will influence the construction.

A viscous fluid model prescribes σ as a function of r, or vice-versa, and implies constant r for constant σ, and constant σ for constant r, in contrast to the variation with time shown in Figs 1 and 2. By allowing stress to depend also on strain-acceleration, so that σ is given as a function of r and $\dot{r}$, the primary creep am in Fig. 1 of a constant stress test can be modelled, as demonstrated by Morland (1979) in the case when the minimum strain-rate is an asymptotic limit; that is, accelerating tertiary creep is absent. However, we will show that such a first order differential relation for r in terms of σ can describe the full constant stress response of Fig. 1. In contrast, the stress variation at constant strain-rate shown in Fig. 2 cannot be described by such a model which reduces simply to a viscous fluid law, σ a function of r, when $\dot{r} = 0$. By analogy with the transient creep construction, it is necessary to incorporate dependence also on stress-rate, so that when $\dot{r} = 0$ the reduced form is a first order differential relation

for σ in terms of r. We will now construct a frame indifferent tensor relation relating stress, stress-rate, strain-rate, strain-acceleration which can model both families of response curves, and which necessarily has the minimal rate-dependence for this purpose. The complete constant stress response can be matched, but completion of the model requires some, but not all, of the constant strain-rate response, or vice-versa. That is, constant stress and constant strain-rate responses reflect some, but not all, of the material properties. We also consider a simpler reduced form, and demonstrate by illustrations the extent to which it reproduces a set of idealised responses.

Differential operator law

The conventional incompressibility approximation is made so that the Cauchy stress $\underset{\sim}{\sigma}$ is determined by the deformation history only within an arbitrary additive isotropic pressure $p\underset{\sim}{1}$. That is, the constitutive law determines only the deviatoric stress

$$\underset{\sim}{S} = \underset{\sim}{\sigma} - \frac{1}{3}\,\mathrm{tr}\,\underset{\sim}{\sigma}\,\underset{\sim}{1}\,, \qquad \mathrm{tr}\,\underset{\sim}{S} = 0\,. \tag{3}$$

The general viscous fluid law can be written (Morland 1979)

$$\underset{\sim}{S} = \phi_1(I_k)\underset{\sim}{D} + \phi_2(I_k)\left[\underset{\sim}{D}^2 - \frac{2}{3}I_2\,\underset{\sim}{1}\right], \tag{4}$$

or equivalently

$$\underset{\sim}{D} = \psi_1(J_k)\underset{\sim}{S} + \psi_2(J_k)\left[\underset{\sim}{S}^2 - \frac{2}{3}J_2\,\underset{\sim}{1}\right], \tag{5}$$

where the strain rate tensor $\underset{\sim}{D}$ is the symmetric part of the spatial velocity gradient, and the non-trivial strain-rate invariants I_k and deviatoric stress invariants J_k $(k = 2,3)$

are

$$I_2 = \frac{1}{2}\,\mathrm{tr}\,\underset{\sim}{D}^2\ ,\quad I_3 = \det \underset{\sim}{D}\ ,\quad J_2 = \frac{1}{2}\,\mathrm{tr}\,\underset{\sim}{S}^2,\quad J_3 = \det \underset{\sim}{S}\ . \qquad (6)$$

The forms (4), (5) are consistent with the requirements $J_1 = \mathrm{tr}\,\underset{\sim}{S} \equiv 0$, $I_1 = \mathrm{tr}\,\underset{\sim}{D} \equiv 0$ (incompressibility), and the response coefficients ϕ_1, ϕ_2 and ψ_1, ψ_2 depend only on two invariants I_2, I_3, and J_2, J_3 respectively. Analytic inversion of the general forms (4), (5) is not possible, but a conventional glaciology flow law assumes $\psi_2 \equiv 0$, $\psi_1 = \psi_1(J_2)$, which implies $\phi_2 \equiv 0$, $\phi_1 = \phi_1(I_2)$ given implicitly by

$$\phi_1(I_2)\ \psi_1[I_2\phi_1^2(I_2)] \equiv 1. \qquad (7)$$

Clearly, uni-axial response cannot determine two functions ϕ_1, ϕ_2, or ψ_1, ψ_2, each depending on two arguments I_2, I_3, or J_2, J_3, respectively,and furthermore, bi-axial tests determine only the same combination $(3I_2)^{\frac{1}{2}}\phi_1 + I_2\phi_2$ on $I_3 = \pm 2(I_2/3)^{\frac{3}{2}}$, or $(3J_2)^{\frac{1}{2}}\psi_1 + J_2\psi_2$ on $J_3 = \pm 2(J_2/3)^{\frac{3}{2}}$, with two independent principal stresses σ_1, $\sigma_2 = \sigma_3$ (Morland 1979). This is a direct consequence of incompressibility $\mathrm{tr}\,\underset{\sim}{D} = 0$ which imposes one relation between the principal strain-rates, $d_1 + 2d_2 = 0$ in bi-axial deformation, and correspondingly $s_1 + 2s_2 = 0$ from $\mathrm{tr}\,\underset{\sim}{S} = 0$, so that each principal component of the law (4) or (5) is a multiple of the others. Thus, tri-axial tests with three independent principal stress components $\sigma_1, \sigma_2, \sigma_3$, or combined axial and shear stress tests (shear stress and shear strain-rate do not enter $\mathrm{tr}\,\underset{\sim}{S}$ and $\mathrm{tr}\,\underset{\sim}{D}$), are required to determine two functions of two arguments. This lack of determinancy

of tensor relation shape by uni-axial data is compounded when strain-acceleration and stress rates are incorporated in the law.

A varying strain-rate $r(t)$ under constant stress, Fig. 1, can be modelled by including stress dependence on the strain-acceleration of a material element, measured by the second order Rivlin-Ericksen tensor

$$\underset{\sim}{A}^{(2)} = 2\,\dot{\underset{\sim}{D}} + 4\,\underset{\sim}{D}^2 + 2(\underset{\sim}{D}\,\underset{\sim}{W} - \underset{\sim}{W}\,\underset{\sim}{D}) \,, \tag{8}$$

where the rotation rate $\underset{\sim}{W}$ is the skew part of the spatial velocity gradient

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W} \,. \tag{9}$$

$\underset{\sim}{A}^{(2)}$ is a frame indifferent tensor. Note that

$$\mathrm{tr}\,\underset{\sim}{A}^{(2)} = 4\mathrm{tr}\,\underset{\sim}{D}^2 = 8I_2 \tag{10}$$

since $\mathrm{tr}\,\underset{\sim}{D} \equiv 0$, $\mathrm{tr}(\underset{\sim}{D}\,\underset{\sim}{W}) = \mathrm{tr}(\underset{\sim}{W}\,\underset{\sim}{D})$, while second and third invariants will depend on $\dot{I}_2$, $\dot{I}_3$. All products of powers of $\underset{\sim}{D}$ with powers of $\underset{\sim}{A}^{(2)}$, with coefficients depending on invariants of $\underset{\sim}{D}$, $\underset{\sim}{A}^{(2)}$ and such products, are frame indifferent, so there is no simple compact frame indifferent expression. Following Morland (1979), the dependence on strain-acceleration will be limited to the first power $\underset{\sim}{A}^{(2)}$, but here, the coefficient is allowed to depend on $\dot{I}_2$ as well as on I_2, I_3, J_2, J_3, to construct one model which describes both primary and tertiary creep.

Similarly, a varying stress $\sigma(t)$ at constant strain-rate, Fig. 2, can be modelled by including a frame indifferent deviatoric

stress rate

$$\underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}} + \underset{\sim}{S}(\underset{\sim}{D} + \underset{\sim}{W}) + (\underset{\sim}{D} - \underset{\sim}{W})\underset{\sim}{S}, \quad \mathrm{tr}\,\underset{\sim}{S}^{(1)} = 2\mathrm{tr}(\underset{\sim}{S}\,\underset{\sim}{D}), \tag{11}$$

with coefficient a function of all invariants, but here limited to I_2, I_3, J_2, J_3, $\dot{J}_2$. Thus, an appropriate frame indifferent relation between $\underset{\sim}{D}$, $\underset{\sim}{A}^{(2)}$, $\underset{\sim}{S}$, $\underset{\sim}{S}^{(1)}$, is

$$\psi_1\underset{\sim}{S} + \psi_3[\underset{\sim}{S}^{(1)} - \tfrac{2}{3}\mathrm{tr}(\underset{\sim}{S}\,\underset{\sim}{D})\underset{\sim}{1}] = \phi_1\underset{\sim}{D} + \phi_2[\underset{\sim}{D}^2 - \tfrac{2}{3}I_2\underset{\sim}{1}] + \phi_3[\dot{\underset{\sim}{D}} + \underset{\sim}{D}\,\underset{\sim}{W} - \underset{\sim}{W}\,\underset{\sim}{D}] \tag{12}$$

where the $\underset{\sim}{D}^2$ term of $\underset{\sim}{A}^{(2)}$ is incorporated in the ϕ_2 term. Note that $\underset{\sim}{A}^{(2)} = \underset{\sim}{0} \not\Leftrightarrow \dot{\underset{\sim}{D}} = \underset{\sim}{0}$, even for $\underset{\sim}{W} = \underset{\sim}{0}$. In one model ϕ_1, ϕ_2, ψ_1 are functions of I_2, I_3, J_2, J_3, but ϕ_3, ψ_3 depend also on $\dot{I}_2$, $\dot{J}_2$, respectively. An alternative dependence on strain-acceleration and stress-rate only through the rate invariants $\dot{I}_2$, $\dot{J}_2$, is obtained by setting $\phi_3 = \psi_3 = 0$ and allowing ψ_1 to depend on the rate invariants. The mixed combinations $\psi_3 = 0$, $\phi_3 \neq 0$, and $\phi_3 = 0$, $\psi_3 \neq 0$ are also considered. Again, uni-axial response cannot determine such a shape distinction.

Uni-axial response

Now consider a uni-axial compressive stress $-\sigma_{11} = \sigma > 0$, other $\sigma_{ij} = 0$, with corresponding strain-rate $-D_{11} = r > 0$, $D_{22} = D_{33} = \frac{1}{2}r$, other $D_{ij} = 0$, and zero rotation-rate $\underset{\sim}{W} \equiv \underset{\sim}{0}$. Then

$$\underset{\sim}{S} = \frac{1}{3}\begin{pmatrix} -2\sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \quad \underset{\sim}{S}^{(1)} = \dot{\underset{\sim}{S}} + \frac{1}{3}\begin{pmatrix} 4\sigma r & 0 & 0 \\ 0 & \sigma r & 0 \\ 0 & 0 & \sigma r \end{pmatrix}, \tag{13}$$

$$\underset{\sim}{D} = \frac{1}{2}\begin{pmatrix} -2r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}, \quad \underset{\sim}{D}^2 = \frac{1}{4}\begin{pmatrix} 4r^2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \quad \underset{\sim}{A}^{(2)} = 2\dot{\underset{\sim}{D}} + 4\underset{\sim}{D}^2, \tag{14}$$

$$J_2 = \frac{1}{3}\sigma^2, \quad J_3 = -\frac{2}{27}\sigma^3 = -2(\tfrac{1}{3}J_2)^{\frac{3}{2}}, \quad \dot{J}_2 = \frac{2}{3}\sigma\dot{\sigma}, \quad \mathrm{tr}\, S^{(1)} = 2r\sigma, \tag{15}$$

$$I_2 = \frac{3}{4}r^2, \quad I_3 = -\frac{1}{4}r^3 = -2(\tfrac{1}{3}I_2)^{\frac{3}{2}}, \quad \dot{I}_2 = \frac{3}{2}r\dot{r}. \tag{16}$$

Each principal component of the law (12) gives

$$\frac{2}{3}\psi_1\sigma + \frac{2}{3}\psi_3(\dot{\sigma} - r\sigma) = \phi_1 r - \frac{1}{2}\phi_2 r^2 + \phi_3\dot{r} \tag{17}$$

where ϕ_1, ϕ_2, ϕ_3 depend on r and $\dot{r}$ through (16), and ψ_1, ψ_3 depend on σ, $\dot{\sigma}$, and r through (15).

For constant stress, $\dot{\sigma} = 0$, (17) reduces to

$$\phi_3\dot{r} + \phi_1 r - \frac{1}{2}\phi_2 r^2 + \frac{2}{3}\psi_3\sigma r = \frac{2}{3}\psi_1\sigma, \tag{18}$$

which is a first order differential equation for $r(t)$ given the constant σ and an initial condition

$$r(0+) = r_o(\sigma). \tag{19}$$

Similarly for constant strain-rate, $\dot{r} = 0$, (17) reduces to

$$\frac{2}{3}\psi_3\dot{\sigma} + \frac{2}{3}\psi_1\sigma - \frac{2}{3}\psi_3 r\sigma = \phi_1 r - \frac{1}{2}\phi_2 r^2 \tag{20}$$

which is a first order differential equation for $\sigma(t)$ given the

constant r and an initial condition

$$\sigma(0+) = 0. \tag{21}$$

The non-linear differential equations (18) and (20) must have unique positive solutions $r(t)$, $\sigma(t)$ in $t > 0$, and for each constant σ and constant r respectively, describe qualitatively at least a class of response curves of the form shown in Fig. 1 and a class of response curves of the form shown in Fig. 2. While it was not anticipated that a first order differential law would match a given family of constant stress responses exactly, we find this can be achieved by appropriate choice of the coefficients in (18). In addition, (20) can then match the family of primary stress increases at constant strain-rates, but predicts the stress relaxation.

A symmetric response in tension requires σ, $\dot{\sigma}$ to change sign with r, $\dot{r}$, which is a restriction on the σ, r dependence of the response functions. Independent compression and tension response will in general require dependence on both even and odd invariants in the uni-axial reductions (15), (16), where $J_3 = +2(\frac{1}{3}J_2)^{\frac{3}{2}}$ and $I_3 = +2(\frac{1}{3}I_2)^{\frac{3}{2}}$ for tension. ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_3 determined as functions of r, σ by correlation with compression data can, in principle, be expressed as functions of I_2 or I_3, or a combination, and J_2 or J_3, or a combination, but different dependences on the invariants will predict different tension responses. Hence, complete knowledge of both tension and compression response provides a restriction on I_2, I_3 and J_2, J_3 dependence. A simple example is the viscous law (4), which becomes $\frac{2}{3}\sigma = \phi_1 r - \frac{1}{2}\phi_2 r^2$, subject to symmetric tension-compression response requiring σ/r invariant. Then the

combination $\phi_1 - \frac{1}{2}r\phi_2$ must be invariant as r changes sign, suggesting that ϕ_1 depends on I_2 and I_3^2 and ϕ_2 is an odd power of I_3 multiplied by a function of I_2 and I_3^2.

This assumes smooth response as $r \to \pm 0$. We now focus on the laws (18) and (20) for uni-axial compression, $\sigma > 0$, $r > 0$.

Constant stress response

An essential feature of the creep curve shown in Fig. 1 is the repeat of the strain-rate values in $r_m(\sigma) < r < r_e(\sigma)$ during tertiary creep $t > t_m(\sigma)$. If we considered $r_e(\sigma) > r_o(\sigma)$, then the duplicated values would be in the primary creep range $r_o(\sigma) > r > r_m(\sigma)$. Thus, the differential equation (18) must describe a double-valued $\dot{r}$ for $r_m(\sigma) < r < r_e(\sigma)$, with $\dot{r} = \dot{r}_- < 0$ during the primary creep, $t < t_m(\sigma)$, and $\dot{r} = \dot{r}_+ > 0$ during the tertiary creep $t > t_m(\sigma)$, and $\dot{r}(t_m) = 0$. The necessary form compatible with (18) is

$$\dot{r}^2 + f(r,\sigma)\dot{r} = F(r,\sigma), \quad r(0) = r_o(\sigma) > 0, \tag{22}$$

which implies

$$2\dot{r} = -f \pm (f^2 + 4F)^{\frac{1}{2}}, \; = 2\begin{pmatrix}\dot{r}_+ \\ \dot{r}_-\end{pmatrix} \; . \tag{23}$$

The relation of f and F to the response coefficients is deferred until the constant strain-rate equation has also been treated.

By (23) we see that for each σ and each r that $F > 0$ is a necessary and sufficient condition for one root $\dot{r}_- < 0$ and

one root $\dot{r}_+ > 0$; hence

$$F(r,\sigma) > 0, \quad r_m(\sigma) < r < r_e(\sigma) \; . \tag{24}$$

For $\dot{r}_- \to 0$ and $\dot{r}_+ \to 0$ as $r \to r_m(\sigma)$, it is necessary that

$$f[r_m(\sigma),\, \sigma] = F[r_m(\sigma),\sigma] = 0 \; , \tag{25}$$

and $\dot{r}$ switches from $\dot{r}_-$ to $\dot{r}_+$ as t passes $t_m(\sigma)$ provided that

$$8\ddot{r} = [1 + f(f^2 + 4F)^{-\frac{1}{2}}]\left\{2\,\frac{\partial f}{\partial r}(f^2 + 4F)^{\frac{1}{2}} + \frac{\partial}{\partial r}(f^2 + 4F)\right\}$$

$$> 0 \quad \text{as} \quad t \to t_m(\sigma)- \; ; \tag{26}$$

that is, if

$$\partial f/\partial r > 0, \quad \partial F/\partial r > 0, \quad \text{at} \quad r = r_m(\sigma). \tag{27}$$

Since $f(r,\sigma)$ and $F(r,\sigma)$ have no role in $r < r_m(\sigma)$, it is convenient to extend them continuously by

$$f(r,\sigma) \equiv F(r,\sigma) \equiv 0, \quad r \leqslant r_m(\sigma). \tag{28}$$

The asymptotic condition $r \to r_e(\sigma)$ as $t \to \infty$ requires $\dot{r}_+ \to 0$ as $r \to r_e(\sigma)$, which implies

$$F[r_e(\sigma),\sigma] = 0, \tag{29}$$

and since there is only primary response in $r > r_e(\sigma)$ we set

$$F(r,\sigma) \equiv 0, \quad r \geqslant r_e(\sigma). \tag{30}$$

By definition, the time to approach $r_e(\sigma)$ in tertiary creep is

unbounded; that is

$$t_m(\sigma) + \int_{r_m(\sigma)}^{r} \frac{2dr'}{[f^2(r',\sigma) + 4F(r',\sigma)]^{\frac{1}{2}} - f(r',\sigma)} \to \infty \text{ as } r \to r_e(\sigma), \qquad (31)$$

while the time $t_m(\sigma)$ to minimum strain-rate is bounded:

$$t_m(\sigma) = \int_{r_m(\sigma)}^{r_o(\sigma)} \frac{2dr'}{[f^2(r',\sigma) + 4F(r',\sigma)]^{\frac{1}{2}} + f(r',\sigma)} < \infty \quad . \qquad (32)$$

Given that $f[r_e(\sigma),\sigma] = -\dot{r}_-$ at $r = r_e(\sigma)$ is not zero and f is analytic there, the unbounded integral (31) requires

$$F(r,\sigma) \lesssim O(r_e - r) \quad \text{as} \quad r \to r_e(\sigma), \qquad (33)$$

and the bounded integral (32) requires

$$f(r,\sigma) + [f^2(r,\sigma) + 4F(r,\sigma)]^{\frac{1}{2}} > O(r - r_m) \quad \text{as} \quad r \to r_m(\sigma), \qquad (34)$$

together satisfied by

$$\begin{aligned} &f = O(r - r_m), \quad F = O(r - r_m) \text{ as } r \to r_m, \\ &F = O(r_e - r) \quad \text{as} \quad r \to r_e \ . \end{aligned} \qquad (35)$$

Note that condition (30) implies the reduction of the differential equation (22) on $r_o(\sigma) > r > r_e(\sigma)$ to

$$\begin{aligned} &\dot{r}\,[\dot{r} + f(r,\sigma)] = 0, \quad r(0) = r_o(\sigma), \\ &\quad \dot{r}_+ \equiv 0, \quad \dot{r}_- = -f \ , \end{aligned} \qquad (36)$$

which permits a solution $r \equiv r_o(\sigma)$. The branch $\dot{r}_- = -f(r,\sigma)$, with $f > 0$, determines the primary creep. At the branch switch

$t_m(\sigma)$, $F = 0$, $\dot{r}_+ = \dot{r}_- = 0$, but $\ddot{r} > 0$, and $\dot{r}_+$ becomes positive.

For numerical correlation with an idealised family of uni-axial response we consider a reduced model involving functions of single argument, which could be expressed in terms of invariants I_2, J_2. Define

$$\mathcal{A}_1 = r^2 - r_m^2(\sigma), \quad \mathcal{A}_2 = [r^2 - r_m^2(\sigma)][r_e^2(\sigma) - r^2] , \tag{37}$$

and restrict r dependence by

$$\begin{aligned} f(r,\sigma) &= q(\sigma)\, \bar{f}(\mathcal{A}_1), \quad \bar{f}(\mathcal{A}_1) \equiv 0 \text{ for } \mathcal{A}_1 \leqslant 0, \\ F(r,\sigma) &= q^2(\sigma)\, \bar{F}(\mathcal{A}_2), \quad \bar{F}(\mathcal{A}_2) \equiv 0 \text{ for } \mathcal{A}_2 \leqslant 0, \end{aligned} \tag{38}$$

which satisfy the requirements (25), (29), and subsidiary conditions (28), (30), identically. The choice of arguments $\mathcal{A}_1$, $\mathcal{A}_2$, is for direct expression in terms of I_2. The conditions (35) become

$$\bar{F}(\mathcal{A}_2) = o(\mathcal{A}_2) \text{ as } \mathcal{A}_2 \to 0, \; \bar{f}(\mathcal{A}_1) = o(\mathcal{A}_1) \text{ as } \mathcal{A}_1 \to 0. \tag{39}$$

As r increases from r_m to $r^* = [\frac{1}{2}(r_m^2 + r_e^2)]^{\frac{1}{2}}$, $\mathcal{A}_2$ increases from zero to $\frac{1}{2}(r_e^2 - r_m^2)$, then decreases to zero as r increases from r^* to r_e. Thus $\mathcal{A}_2(r)$ has duplicated values in $r_m \leqslant r \leqslant r_e$, which will restrict correlation over the full primary-tertiary creep response. As r increases from r_m to r_o, $\mathcal{A}_1$ is strictly increasing. With the forms (38),

$$t_m(\sigma) = \frac{1}{q(\sigma)} \int_{r_m(\sigma)}^{r_o(\sigma)} \frac{2dr'}{\left[\bar{f}^2(\mathcal{Q}_1') + 4\bar{F}(\mathcal{Q}_2')\right]^{\frac{1}{2}} + \bar{f}(\mathcal{Q}_1')} , \tag{40}$$

$$\mathcal{Q}_1' = r'^2 - r_m^2 \, , \quad \mathcal{Q}_2' = (r'^2 - r_m^2)(r_e^2 - r'^2) \, .$$

Constant strain-rate response

Analogous to the constant stress analysis, the necessary form compatible with (20) to describe $\sigma(t)$ at constant r shown in Fig. 2 is

$$\dot{\sigma}^2 - g(r,\sigma)\dot{\sigma} = G(r,\sigma), \quad \sigma(0) = 0 \, , \tag{41}$$

which implies

$$2\dot{\sigma} = g \pm (g^2 + 4G)^{\frac{1}{2}}, = 2\begin{Bmatrix} \dot{\sigma}_+ \\ \dot{\sigma}_- \end{Bmatrix} . \tag{42}$$

For $\dot{\sigma}_+ > 0$ and $\dot{\sigma}_- < 0$ on the duplicated stress range we require

$$G(r,\sigma) > 0, \qquad \sigma_E(r) < \sigma < \sigma_M(r) \, , \tag{43}$$

while $\dot{\sigma}_+ \to 0$ and $\dot{\sigma}_- \to 0$ as $\sigma \to \sigma_M(r)$ requires

$$g\left[r,\sigma_M(r)\right] = G\left[r,\sigma_M(r)\right] = 0. \tag{44}$$

$\dot{\sigma}$ switches from $\dot{\sigma}_+$ to $\dot{\sigma}_-$ as t passes through $t_M(r)$ provided that $\ddot{\sigma} < 0$ there; that is, if

$$\frac{\partial g}{\partial r} > 0 \, , \quad \frac{\partial G}{\partial r} > 0 \quad \text{at} \quad \sigma = \sigma_M(r). \tag{45}$$

The asymptotic condition $\sigma \to \sigma_E(r)$ as $t \to \infty$ requires $\dot{\sigma}_- \to 0$ as $\sigma \to \sigma_E(r)$, which implies

$$G[r, \sigma_E(r)] = 0 , \tag{46}$$

and an infinite relaxation time requires, given $g[r, \sigma_E(r)] = \dot{\sigma}_+$ at $\sigma = \sigma_E(r)$ is not zero and g is analytic there,

$$G(r,\sigma) \leqslant O(\sigma - \sigma_E) \quad \text{as} \quad \sigma \to \sigma_E(r). \tag{47}$$

The time to maximum stress is

$$t_M(r) = \int_0^{\sigma_M(r)} \frac{2d\sigma'}{[g^2(r,\sigma') + 4G(r,\sigma')]^{\frac{1}{2}} + g(r,\sigma')} , < \infty \tag{48}$$

provided that

$$g(r,\sigma) + [g^2(r,\sigma) + 4G(r,\sigma)]^{\frac{1}{2}} > O(\sigma_M - \sigma) \text{ as } \sigma \to \sigma_M(r). \tag{49}$$

Conditions (47) and (49) are satisfied by

$$\begin{aligned} &g = O(\sigma_M - \sigma),\ G = O(\sigma_M - \sigma) \text{ as } \sigma \to \sigma_M, \\ &G = O(\sigma - \sigma_E) \quad \text{as} \quad \sigma \to \sigma_E \ . \end{aligned} \tag{50}$$

We can set

$$\begin{aligned} &g(r,\sigma) \equiv G(r,\sigma) \equiv 0, \quad \sigma \geqslant \sigma_M(r), \\ &G(r,\sigma) \equiv 0, \quad \sigma \leqslant \sigma_E(r). \end{aligned} \tag{51}$$

Now on $0 \leqslant \sigma \leqslant \sigma_E(r)$, (41) becomes

$$\begin{aligned} &\dot{\sigma}[\dot{\sigma} - g(r,\sigma)] = 0, \quad \sigma(0) = 0, \\ &\dot{\sigma}_+ = g, \quad \dot{\sigma}_- = 0 , \end{aligned} \tag{52}$$

which allows a solution $\sigma \equiv 0$. However, the primary response is given by $\dot{\sigma}_+ = g(r,\sigma)$, and at the branch switch $t_M(r)$ where $G = 0$, $\dot{\sigma} = 0$, $\dot{\sigma}$ becomes negative since $\ddot{\sigma} < 0$.

Now we apply the inverse relations (1) and (2) which state that the curves $r = r_m(\sigma)$ and $\sigma = \sigma_M(r)$ in an (r,σ) plane are identical, and the curves $r = r_e(\sigma)$ and $\sigma = \sigma_E(r)$ are identical. Furthermore, $r < r_m(\sigma)$, or $\mathcal{a}_1 < 0$, corresponds to $\sigma > \sigma_M(r)$, and $r > r_e(\sigma)$, or $\mathcal{a}_1 > 0$ and $\mathcal{a}_2 < 0$, corresponds to $\sigma < \sigma_E(r)$. Thus the conditions (51) become

$$\begin{aligned} g(r,\sigma) \equiv G(r,\sigma) \equiv 0, \quad & r \leqslant r_m(\sigma), \\ G(r,\sigma) \equiv 0, \quad & r \geqslant r_e(\sigma). \end{aligned} \tag{53}$$

Conditions (28), (30), (53) are illustrated in Fig. 3.

Corresponding to the reduced model (36), (37) we define

$$\begin{aligned} g(r,\sigma) &= s(r)q(\sigma)\bar{g}(\mathcal{a}_1), \quad \bar{g}(\mathcal{a}_1) \equiv 0 \text{ for } \mathcal{a}_1 \leqslant 0, \\ G(r,\sigma) &= q^2(\sigma)\bar{G}(\mathcal{a}_2), \quad \bar{G}(\mathcal{a}_2) \equiv 0 \text{ for } \mathcal{a}_2 \leqslant 0, \\ \bar{G}(\mathcal{a}_2) &= o(\mathcal{a}_2) \text{ as } \mathcal{a}_2 \to 0, \quad \bar{g}(\mathcal{a}_1) = o(\mathcal{a}_1) \text{ as } \mathcal{a}_1 \to 0, \end{aligned} \tag{54}$$

which satisfy the requirements (53) identically, and provided that $r_m'(\sigma)$, $\sigma_M'(r) \neq 0$, satisfy condition (50). It is found that the extra factor $s(r)$, here, or as an f factor in (38), is necessary for time scale control.

<u>Response coefficients</u>

We now construct four sets of response coefficients ψ_1, ψ_3, ϕ_1, ϕ_2, ϕ_3 which reduce the constant stress and constant strain-rate relations (18) and (20) to the required forms (22) and (41). The first supposes that ϕ_3, $\psi_3 \neq 0$ so that there is rate dependence

through both tensors $\underset{\sim}{A}^{(2)}$ and $\underset{\sim}{S}^{(1)}$, and the second assumes $\phi_3 = \psi_3 = 0$ so that rate dependence is through $\dot{I}_2$, $\dot{J}_2$ only. The other two are the combinations $\psi_3 = 0$, $\phi_3 \neq 0$, and $\phi_3 = 0$, $\psi_3 \neq 0$. With each set there is the implication $F(r,\sigma) = G(r,\sigma)$, shown in Fig. 3, which represents the coupling of constant stress and constant strain-rate responses in this model, or the extent to which they hinge on a common material property.

In the case $\phi_3 \neq 0$, $\psi_3 \neq 0$ we can take $\psi_1 = 1$ without loss of generality. Then, restricting rate dependence to the new ϕ_3, ψ_3 and comparing (18) and (22), shows that

$$\phi_3 = \dot{r} + f(r,\sigma), \quad = f(r,\sigma) \quad \text{when} \quad \dot{r} = 0, \tag{55}$$

so ϕ_3 is a function of $\dot{I}_2$, I_2, J_2, I_3, J_3 in general. Comparing (20) and (40) we find ψ_3 is a function of $\dot{J}_2$, I_2, J_2, I_3, J_3 given by

$$\tfrac{2}{3}\psi_3 = -\dot{\sigma} + g(r,\sigma) - r\sigma, \quad = g(r,\sigma) - r\sigma \quad \text{when } \dot{\sigma} = 0\,; \tag{56}$$

this sign choice gives $G = +F$. Then (2), (41) are recovered with

$$F(r,\sigma) = G(r,\sigma) = -\phi_1 r + \tfrac{1}{2}\phi_2 r^2 - r\sigma g(r,\sigma) + r^2\sigma^2 + \tfrac{2}{3}\sigma, \tag{57}$$

so that ϕ_1, ϕ_2 are functions of I_2, J_2, I_3, J_3 only. Again, only a combination of ϕ_1, ϕ_2 is given by F.

In the case $\phi_3 = \psi_3 = 0$, restricting rate dependence to ψ_1, the above comparisons show that

$$\begin{aligned} \tfrac{2}{3}\psi_1\sigma &= -\dot{r}^2 - f(r,\sigma)\dot{r} - \dot{\sigma}^2 + g(r,\sigma)\dot{\sigma} + \mathcal{A}(r,\sigma), \\ F(r,\sigma) &= G(r,\sigma) = -\phi_1 r + \tfrac{1}{2}\phi_2 r^2 + \mathcal{A}(r,\sigma), \end{aligned} \tag{58}$$

where ψ_1 depends on $\dot{I}_2$, $\dot{J}_2$, I_2, J_2, I_3, J_3, and a combination of ϕ_1 and ϕ_2 depends on I_2, J_2, I_3, J_3. Normalising by $\psi_1 = 1$ changes ϕ_1, ϕ_2 but reproduces the tensor law (12) with coefficients given by (58). The arbitrary $\mathcal{a}(r,\sigma)$ represents in (12) an $\underset{\sim}{S}$ term with rate-independent coefficient, with corresponding $\underset{\sim}{D}$, $\underset{\sim}{D}^2$ terms. This generality is lost in the first construction where rate-dependence is restricted to ϕ_3, ψ_3 after setting $\psi_1 = 1$. Clearly, the uni-axial response functions f, g, F cannot determine $\mathcal{a}(r,\sigma)$ in this model. Alternative models with ϕ_1, ϕ_2 rate dependent can be constructed.

Similarly, again excluding rate-dependence from ϕ_1, ϕ_2, for $\phi_3 \neq 0$, $\psi_3 = 0$,

$$\begin{aligned} \phi_3 &= \dot{r} + f(r,\sigma), \\ \tfrac{2}{3}\psi_1\sigma &= -\dot{\sigma}^2 + g(r,\sigma)\dot{\sigma} + b(r,\sigma) , \end{aligned} \tag{59}$$

$$F(r,\sigma) = G(r,\sigma) = -\phi_1 r + \tfrac{1}{2}\phi_2 r^2 + b(r,\sigma),$$

and for $\phi_3 = 0$, $\psi_3 \neq 0$,

$$\begin{aligned} \tfrac{2}{3}\psi_3 &= -\dot{\sigma} + g(r,\sigma) - r\sigma , \\ \tfrac{2}{3}\psi_1\sigma &= -\dot{r}^2 - f(r,\sigma)\dot{r} + c(r,\sigma), \end{aligned} \tag{60}$$

$$F(r,\sigma) = G(r,\sigma) = -\phi_1 r + \tfrac{1}{2}\phi_2 r^2 + r^2\sigma^2 - r\sigma g(r,\sigma) + c(r,\sigma)$$

where $b(r,\sigma)$, $c(r,\sigma)$ are arbitrary.

The variety of dependence of the response coefficients on r and σ eliminates simple dependence on integral powers of I_2, J_2, but including I_3, J_3 allows rational functions of integral powers. Dependence on $\mathrm{tr}(\underset{\sim}{S}\,\underset{\sim}{D}) = r\sigma$ could also be incorporated. Uni-axial response does not distinguish the various forms of

dependence. Each of the four models described above will predict different response in any non-uni-axial stress geometry, for example, simple shear stress or simple shear motion, and different dependence on invariants within each model similarly predicts different response. Finally, following Morland (1979), a tentative initial condition for the tensor law (12) when stress is prescribed is

$$\underset{\sim}{D}(0) = h(J_2)\underset{\sim}{S}(0), \quad h(\tfrac{1}{3}\sigma^2) = \frac{3}{2}\,\frac{r_o(\sigma)}{\sigma}. \tag{61}$$

Uni-axial response functions

It is evident that three functions $f(r,\sigma)$, $g(r,\sigma)$, $F(r,\sigma)$ cannot exactly reproduce independent strain-rate curves $r(t)$ for each constant σ and stress curves $\sigma(t)$ for each constant r. A differential operator law including only strain-acceleration and strain-rate evaluated at current time is a restricted history dependence. We can incorporate the important properties $r_m(\sigma)$, $r_e(\sigma)$ through the $\mathcal{A}_1$, $\mathcal{A}_2$ definitions (37), and the associated properties $\sigma_M(r)$, $\sigma_E(\sigma)$ through the inverse relations (1). Because of the restriction $F = G$ there are two functions f, F, available to correlate with constant stress response, but then only one function is available to correlate with constant strain-rate response, or vice-versa. Thus one class of response is given priority. We will develop the former case, but a similar procedure applies to the latter.

For each constant σ the given strain-rate response determines $\dot{r}_+(t) = R_+(r,\sigma) \geqslant 0$ and $\dot{r}_-(t) = R_-(r,\sigma) \leqslant 0$. By (23),

$$r_m \leqslant r \leqslant r_o: \quad f = -(R_+ + R_-), \quad (f^2 + 4F)^{\frac{1}{2}} = R_+ - R_-, \tag{62}$$

with $F \equiv 0$ and $R_+ \equiv 0$ for $r_e \leqslant r \leqslant r_o$. Recalling the sufficient analytic requirement (35), we can write

$$f = \mathcal{a}_1 \hat{f} \quad , \quad F = \mathcal{a}_2 \hat{F} \, , \tag{63}$$

where $\hat{f}$, $\hat{F}$ are non-zero at $r = r_m(\sigma)$ and $r = r_e(\sigma)$, and factor $\mathcal{a}_1$ from $(R_+ + R_-)$ data near $r = r_m(\sigma)$, and $\mathcal{a}_2^{\frac{1}{2}}$ from $(R_+ - R_-)$ data near $r = r_m(\sigma)$ and $r = r_e(\sigma)$. f, $\hat{f}$ are given directly by $(62)_1$ and then F, $\hat{F}$ by $(62)_2$, and solution over a maximum (r,σ) domain allows complete reproduction of constant stress response by the differential law.

In the reduced model (37), (38), the argument $\mathcal{a}_2$ of $\bar{F}$ at a given constant σ is duplicated in the intervals $r_m \leqslant r \leqslant r^*$ and $r^* \leqslant r \leqslant r_e$ where

$$r^* = \left[\frac{1}{2}(r_m^2 + r_e^2)\right]^{\frac{1}{2}}. \tag{64}$$

Thus the relation $(62)_2$ can be used in only one of these intervals. We apply (62) on the intervals $r_m \leqslant r \leqslant r^*$, $r_e \leqslant r \leqslant r_o$, so that F is fully determined, and $(23)_2$ on $r^* \leqslant r \leqslant r_e$ to complete f. Hence primary creep is matched over the complete range $r_m \leqslant r < r_o$, but tertiary creep only over $r_m \leqslant r \leqslant r^*$, so that the reduced model predicts tertiary creep over $r^* \leqslant r \leqslant r_e$, continuous with data at r^* and r_e. The important primary-tertiary transition is matched. This correlation has determined $f = q(\sigma)\bar{f}(\mathcal{a}_1)$ and $F = q^2(\sigma)\bar{F}(\mathcal{a}_2)$ on $0 < \mathcal{a}_1 < r_o^2(\sigma) - r_m^2(\sigma)$ and $0 < \mathcal{a}_2 < \frac{1}{2}\left[r_e^2(\sigma) - r_m^2(\sigma)\right]$. As the constant σ is varied there will be a common range of $\mathcal{a}_1$ and of $\mathcal{a}_2$ over which $\bar{f}$ and $\bar{F}$ are already determined, and hence f and F within the

factors $q(\sigma)$, $q^2(\sigma)$ respectively. But independent response $R_+(r,\sigma)$, $R_-(r,\sigma)$ at a different σ will require $f(r,\sigma)$ satisfying (62) with r dependence in general not compatible with the above functions. Thus the restricted r dependence of (38) allows the detailed time response at one constant σ only to be described, but does incorporate the properties $r_m(\sigma)$, $r_e(\sigma)$, and $r_o(\sigma)$, for all σ. We suppose $r_m(\sigma)$, $r_e(\sigma)$, $r_o(\sigma)$ all increase with σ, and also that $r_o - r_m$ and $r^* - r_m$ increase so that the ranges of both $\mathcal{a}_1$ and $\mathcal{a}_2$ increase as σ increases. Hence we correlate the time response for the maximum σ, $= \Sigma$ say. Setting $q(\Sigma) = 1$, $\bar{f}(\mathcal{a}_1)$ and $\bar{F}(\mathcal{a}_2)$ are now fully determined, and the factor $q(\sigma)$ allows one more feature of the constant $\sigma(<\Sigma)$ curves to be reproduced.

For illustrations we prescribe in turn $t_m(\sigma)$, related to $q(\sigma)$ by (40), and $\varepsilon_m(\sigma) = 0{\cdot}01$ where

$$\varepsilon_m = e^{\bar{\varepsilon}_m} - 1, \quad \bar{\varepsilon}_m(\sigma) = \int_0^{t_m(\sigma)} r(t)dt. \tag{65}$$

By (23), (38) $\dot{r}_- = -q(\sigma)\{\bar{f}(\mathcal{a}_1) + [\bar{f}^2(\mathcal{a}_1) + 4\bar{F}(\mathcal{a}_2)]^{\frac{1}{2}}\}$, which has a solution for each constant σ of the form $r_-(t) = \theta[q(\sigma)t,\sigma]$ where $\theta(0,\sigma) = r_o(\sigma)$ and $\theta(t^*,\sigma)$ is determined by $\bar{f}$ and $\bar{F}$, independent of $q(\sigma)$. By (40), $t_m^*(\sigma) = q(\sigma)t_m(\sigma)$ is determined by $\bar{f}$, $\bar{F}$, independent of $q(\sigma)$, so changing variable in (65) to $t^* = q(\sigma)t$ gives

$$\bar{\varepsilon} = \frac{1}{q(\sigma)} \int_0^{t_m^*(\sigma)} \theta[t^*,\sigma]dt^* . \tag{66}$$

Thus $q(\sigma)$ is determined by prescribing $\varepsilon_m(\sigma)$, hence $\bar{\varepsilon}$,

since the integral is independent of $q(\sigma)$.

For each constant r the given stress response determines $\dot{\sigma}_+(t) = S_+(r,\sigma) \geqslant 0$ and $\dot{\sigma}_-(t) = S_-(r, \ ) \leqslant 0$, and by (42)

$$0 \leqslant \sigma \leqslant \sigma_M: \quad g = S_+ + S_-, \quad (g^2 + 4F)^{\frac{1}{2}} = S_+ = S_- , \tag{67}$$

with $F \equiv 0$ and $S_- \equiv 0$ for $0 \leqslant \sigma \leqslant \sigma_E$ $(r \geqslant r_e)$. Both relations (67) cannot be satisfied since F has been determined by constant stress response. This is the extent to which constant σ and constant r response reflect a common material property under this model. g can be calculated by either $(67)_1$ or $(67)_2$, when the other fails in general, or, perhaps better, the complete initial response OM (Fig. 2) can be reproduced by matching $\dot{\sigma}_+$ from (42):

$$g + (g^2 + 4F)^{\frac{1}{2}} = 2S_+ . \tag{68}$$

The time variation $\sigma(t)$ during relaxation ME (Fig. 2) is then predicted, but $t_M(r)$, $\sigma_E(r)$ match given data.

In the reduced model (54), $G = F$ and q are already known, leaving $\bar{g}(\mathcal{a}_1)$ and $s(r)$ free. Setting $s = 1$ at the maximum strain-rate $r_o(\Sigma)$, $\bar{g}(\mathcal{a}_1)$ can be calculated by (66) to reproduce the initial response OM at $r_o(\Sigma)$. Then for $r_m(\Sigma) \leqslant r \leqslant r_o(\Sigma)$, $s(r)$ can be determined to match prescribed times $t_M(r)$, for which (48) becomes

$$t_M(r) = \int_o^{\sigma_M(r)} \frac{2d\sigma'}{q(\sigma')\{[s^2(r)\bar{g}^2(\mathcal{a}_1'') + 4\bar{F}(\mathcal{a}_2'')]^{\frac{1}{2}} + s(r)g(\mathcal{a}_1'')\}} , \tag{69}$$

where

$$\mathcal{a}_1'' = r^2 - r_m^2(\sigma'), \quad \mathcal{a}_2'' = [r^2 - r_m^2(\sigma')][r_e^2(\sigma') - r^2], \tag{70}$$

or to match the strain at maximum stress $\varepsilon_M(r) = \varepsilon[\sigma_M(r)]$. The integral is evaluated for a sequence of $s(r)$, each r, until $t_M(r)$ is attained. In our illustrations, since we cannot construct constant strain-rate responses which correspond to the idealised constant stress responses, we set $\bar{g} = \bar{f}$ and consider both $t_M(r) = t_m[\sigma_M(r)]$ and $\varepsilon_M(r) = 0{\cdot}01$, that is $\bar{\varepsilon}_M = \ell n(1{\cdot}01)$.

Numerical evaluation of the complete functions $f(r,\sigma)$, $g(r,\sigma)$, $F(r,\sigma)$, by (62), (63) and (67) or (68) in the dependence domain of the (r,σ) plane, shown in Fig. 3, is straightforward, but the large matrices of tabulated values are worthwhile only when a full set of good constant stress and strain-rate data is available.

Idealised response and model correlation

To demonstrate the construction of the simplified reduced model (37), (38), (54), and the degree to which it describes the complete constant stress response, we have treated two different families of idealised curves of the shape shown in Fig. 1. Each has the form

$$r = r_e(\sigma) - Be^{-bt} + Ce^{-ct}, \tag{71}$$

where the stress dependent positive parameters B, b, C, c are chosen to match given $r_o(\sigma)$, $r_m(\sigma)$, $\tau = t_i/t_m$ where t_i

is the time to inflexion $(\ddot{r} = 0)$ on the tertiary creep curve me, and $t_m(\sigma)$ or $\varepsilon_m(\sigma)$. In each family the Colbeck and Evans (1973) polynomial flow law near melting is adopted for the minimum strain-rate:

$$r_m = 0{\cdot}21\sigma + 0{\cdot}14\sigma^3 + 0{\cdot}055\sigma^5, \tag{72}$$

where σ is measured in bars $(10^5 Nm^{-2})$ and strain-rate in years^{-1}.

In family 1:

$$r_e = r_m(1 + \sigma^{\frac{1}{2}}), \quad r_o = r_m(1 + 2\sigma^{\frac{1}{2}}), \quad t_m = t_1\sigma^{-\frac{1}{3}}, \quad \tau = 1{\cdot}8, \tag{73}$$

where t_1 is value of t_m at $\sigma = 1$ bar. To be specific we adopt $t_1 = 15$ hours, within the range indicated by Colbeck and Evans (1973). Choosing the maximum stress $\Sigma = 2$ bars, with $q(\Sigma) = 1$, the relations (62) determine $\bar{f}(\mathcal{a}_1)$, $\bar{F}(\mathcal{a}_2)$, and $q(\sigma)$ for $\sigma < \Sigma$ is determined by the t_m relation (40). Two constant strain-rate responses are modelled by setting $\bar{g}(\mathcal{a}_1) = \bar{f}(\mathcal{a}_1)$ and determining $s(r) = s_1(r)$ and $s(r) = s_2(r)$ in the g decomposition (54) from (69) with $t_M(r) = t_m[\sigma_M(r)] = t_m[r_m^{-1}(r)]$ by (2), and $t_M(r) = \bar{\varepsilon}[\sigma_M(r)]/r$, respectively. The functions $\bar{f}(\mathcal{a}_1)$, $\bar{F}(\mathcal{a}_2)$, $1/q(\sigma)$, $1/s_1(r)$, $1/s_2(r)$, are shown in Fig. 4. A selection of constant stress responses $r(t)$ at $\sigma = \sigma_j$, given by the idealised functions (72), are shown by the solid lines in Fig. 5, and the corresponding responses predicted by the differential operator model (22) are shown by the dashed lines. By construction the

$\sigma = \Sigma = 2$ bars curves match identically expect on the final stage $r^* < r < r_e$ of the tertiary creep, and r_o, r_e, r_m, t_m match for all σ. The agreement between the reduced model and the parent idealised response is excellent. Figure 6 shows the corresponding strain responses $\varepsilon(t)$ calculated from the given strain-rates and predicted strain-rates. Stress-strain curves at constant strain-rates $\dot{r} = r_m(\sigma_j)$ are shown in Fig. 7 for $s = s_1$ (———) and $s = s_2$ (-------). Note that a differential operator law which predicts primary creep at constant stress with monotonic $\dot{r}$ still allows a primary stress-strain (time) response at constant r with an inflexion point. The assumption $\bar{g} = \bar{f}$ is of course artificial, and matching with given constant strain-rate response may yield a different picture.

In family 2:

$$r_e = r_m[1 + \theta(\sigma)], \quad r_o = r_m[1 + k\theta(\sigma)], \quad t_m = t_1\sigma^{-1}, \tau = 1{\cdot}5,$$
$$k = \frac{r_o - r_m}{r_e - r_m} = 11\ , \tag{74}$$

and $\theta(\sigma)$ is determined by setting $\varepsilon_m = 0{\cdot}01$ in (65). Again $\bar{f}$ and $\bar{F}$ are determined by matching the $\sigma = \Sigma = 2$ bars response except for tertiary creep on $r^* < r < r_e$. First we determine $q(\sigma) = q_1(\sigma)$ by matching the prescribed $t_m(\sigma)$ for $\sigma < \Sigma$ through the relation (40), and find as expected that the independent property $\varepsilon_m(\sigma) = \varepsilon_c(\sigma)$, calculated from the predicted $r(t)$, is not $0{\cdot}01$. The comparison of prescribed (—) and predicted (----) $r(t)$ curves at selected stresses $\sigma = \sigma_j$ is

shown in Fig. 8, and the comparison of strains $\varepsilon(t)$ determined by the prescribed and predicted strain-rates is shown in Fig. 9, highlighting the differences in given ε_m and predicted $\varepsilon_m = \varepsilon_c$. Next we determine $q = q_2(\sigma)$ for the predicted strain $\varepsilon_m(\sigma)$ at minimum strain rate, given by (65), (66), to be 0·01, and show the corresponding $r(t)$ and $\varepsilon(t)$ curves (....) in Figs 8 and 9. Now the predicted $t_m(\sigma_j)$ do not match the prescribed $t_m(\sigma_j)$. The primary creep prediction appears better for the choice $q = q_2$, but the overall match looks better with $q = q_1$. Matching of the reduced model prediction with the given response is clearly less satisfactory for family 2 than family 1. Similar discrepancies have been obtained with $\tau = 1{\cdot}6$ and $k = 5$ in (74). A distinct difference between the two families is that $\theta = \sigma^{\frac{1}{4}} \to 0$ as $\sigma \to 0$ in family 1, while $\theta(\sigma) \to \theta_0 > 0$ as $\sigma \to 0$ in family 2, and while the response functions $\bar{f}$, $\bar{F}$, s are similar for the two families, illustrated in Fig. 4, $1/q \to 0$ as $\sigma \to 0$ in family 1, but not in family 2.

Constant strain-rate responses at $r = r_m(\sigma_j)$ are constructed by determining (i) $s = s_1(r)$ for $t_M(r) = t_m[r_m^{-1}(r)]$, (ii) $s = s_2(r)$ for $t_M(r) = \bar{\varepsilon}[\sigma_M(r)]/r$, (iii) $s = s_3(r)$ for $t_M(r) = 0{\cdot}01/r$, all with $q = q_1(\sigma)$. These are illustrated by the stress-strain curves shown (——), (-----), (.....) respectively in Fig. 10. The significant peaks as r decreases, that is, large σ_M/σ_E. arise because $r_e/r_m \to 1$ as $\sigma \to 0$ in family 1, but approaches 11·45 in family 2.

Section 2. Viscoelastic solid laws of differential type

Introduction

The viscoelastic fluid law described in section 1 is necessarily isotropic in all configurations, and to describe anisotropy, whether in an initial reference configuration or in a subsequent configuration reached by loading, a solid law is required with explicit dependence on a reference configuration. No simple non-linear viscoelastic solid laws have been constructed or analysed in other fields of rheology, so we adopt a heuristic construction by analogy with the successful fluid law of differential type. Eventually we hope to construct an integral operator law which allows more general dependence on deformation history, and which should be more satisfactory for numerical computation. A preliminary examination of an integral law linear in finite strain shows this cannot describe important features of ice response, so a new approach must be investigated.

Mellor (1980) notes that the typical constant stress and constant strain-rate responses shown in Figs 1 and 2 are often constant load and constant displacement-rate responses, at least for small strains achieved in laboratory tests. This is an immediate reference to the initial configuration. For uni-axial compressive stress $\sigma > 0$ there is an axial stretch $\lambda < 1$ (contraction) and equal lateral stretches $\lambda^{-\frac{1}{2}}$ for incompressible material. The deformation gradient and associated Cauchy-Green strain tensors are

$$\underset{\sim}{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-\frac{1}{2}} & 0 \\ 0 & 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}, \quad \underset{\sim}{B} = \underset{\sim}{C} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \tag{2.1}$$

with strain-invariants $\det \underset{\sim}{B} \equiv 1$ and

$$K_1 = \mathrm{tr}\underset{\sim}{B} = \lambda^2 + 2\lambda^{-1}, \quad K_2 = \tfrac{1}{2}\{(\mathrm{tr}\underset{\sim}{B})^2 - \mathrm{tr}\underset{\sim}{B}^2\} = 2\lambda + \lambda^{-2}. \tag{2.2}$$

The axial compressive (engineering) strain, contraction per unit initial length, is

$$e = 1 - \lambda, \tag{2.3}$$

and the compressive strain-rate relative to the current configuration is

$$r = -\lambda^{-1}\dot{\lambda} = (1 - e)^{-1}\dot{e}. \tag{2.4}$$

Thus constant displacement-rate, which implies constant $\dot{\lambda}$ and $\dot{e}$, implies constant $r\lambda$. In particular, an asymptotic strain-rate $r_e > 0$ implies that $\lambda \to 0$ as $\dot{\lambda} \to 0$, $e \to 1$ as $\dot{e} \to 0$. With small instantaneous elastic strain e_e when a load is applied and maintained, the typical monotonic $e - t$ curve is shown in Fig. 11, indicating a 1:1 relation so that the engineering strain-rate $\dot{e}$ may be expressed as a function of e as shown in Fig. 12. The dashed sections indicate the assumed finite strain response consistent with the above asymptotic behaviour, but we are mainly concerned with a model which describes the small strain response, before a maximum strain-rate and subsequent strain-rate decrease are reached. Mellor suggests

that e_m, corresponding to minimum strain-rate $\dot{e}_m = (1 - e_m)r_m$, is approximately 0·01 over a wide range of applied stress.

Force per unit initial (reference) area is measured by the nominal stress $\bar{\sigma} = F^{-1}\sigma$ (det $F \equiv 1$), so that constant load is equivalent to constant $\bar{\sigma}$. In uni-axial compressive stress, with $\bar{\sigma}_{11} = -\bar{\sigma}$,

$$\bar{\sigma} = \lambda^{-1}\sigma \ , \quad \dot{\sigma} = \lambda\dot{\bar{\sigma}} + \dot{\lambda}\bar{\sigma} \quad . \qquad (2.5)$$

At constant displacement rate, constant $\dot{e}$, the typical stress-strain curve is shown in Fig. 13, with the dashed section indicating response beyond small strain observations in the laboratory. Stress increases to a maximum $\bar{\sigma}_M$ at strain e_M which is also expected to be approximately 0·01 for a wide range of $\dot{e}$, then relaxes, but as compression increases there must be a minimum load followed by unbounded increase.

The most simple stress dependence on a reference configuration is through a frame indifferent tensor function of deformation gradient F. Then in uni-axial stress, $\bar{\sigma}$ depends explicitly on e. In the fluid model response to constant stress, (18) of section 1, it was necessary to incorporate a strain-acceleration term $\dot{r}$ to exhibit the time variation $r(t)$ shown in Fig. 1. Here, however, dependence on strain-rate D, and hence on $\dot{e}$, is sufficient, since an $\dot{e}$, e relation allows the strain-rate variation shown in Fig. 12. The 1:1 $e - t$ relation shown

in Fig. 11 replaces the time variable by strain, but the differential fluid model has no explicit dependence on strain and (18) gives constant r at constant σ if the $\dot{r}$ term is absent. Again we require a stress-rate term to exhibit a time variation $\bar{\sigma}(t)$, and hence $\bar{\sigma}(e)$ variation shown in Fig. 13 at constant displacement rate. By analogy with (12) in section (1), an appropriate frame-indifferent differential incompressible solid law is

$$\underset{\sim}{S} + \psi_3[\underset{\sim}{S}^{(1)} - \tfrac{2}{3}\mathrm{tr}(\underset{\sim}{S}\,\underset{\sim}{D})\underset{\sim}{1}] =$$
$$[\underset{\sim}{F}\,\underset{\sim}{f}(\underset{\sim}{C})\underset{\sim}{F}^T - \tfrac{1}{3}\mathrm{tr}(\underset{\sim}{C}\,\underset{\sim}{f})\underset{\sim}{1}] + \phi_1\underset{\sim}{D} + \phi_2[\underset{\sim}{D}^2 - \tfrac{2}{3}I_2\underset{\sim}{1}] \;, \qquad (2.6)$$

where $\underset{\sim}{f}$ is an arbitrary symmetric tensor function of symmetric tensor argument, and ψ_3, ϕ_1, ϕ_2 are functions of any invariants. We have normalised with unit stress deviator coefficient, and assumed linearity in the stress tensor rate $\underset{\sim}{S}^{(1)}$.

If the solid is isotropic in the reference configuration then

$$\underset{\sim}{F}\,\underset{\sim}{f}(\underset{\sim}{C})\underset{\sim}{F}^T - \tfrac{1}{3}\mathrm{tr}(\underset{\sim}{C}\,\underset{\sim}{f}) = \omega_1[\underset{\sim}{B} - \tfrac{1}{3}K_1\underset{\sim}{1}] + \omega_2[\underset{\sim}{B}^2 - \tfrac{1}{3}(K_1^2 - 2K_2)\underset{\sim}{1}], \qquad (2.7)$$

where ω_1, ω_2 are functions of the invariants K_1, K_2, and possibly of J_2, J_3, and the rate invariants. Explicit dependence on e in uni-axial response can be retained by including dependence of ψ_3, ϕ_1, ϕ_2 on K_1 and K_2 with

$\omega_1 = \omega_2 = 0$; that is, with no dependence on the deformation orientation. The resulting response will depend on the amount of deformation from a reference configuration, but can exhibit no anisotropy. We will adopt the isotropic form (2.7) with (2.6) to show compatibility with uni-axial response, and we plan to investigate a particular form to demonstrate anisotropy in configurations reached by simple loading paths. Initial anisotropy in the reference configuration arising from the ice formation will require a different shape of tensor function $f(\underset{\sim}{C})$.

Uni-axial response and small strain approximation

In uni-axial stress described by the expressions (13) - (16) of section 1, and (2.1) - (2.5), each principal component of (2.6), (2.7) gives

$$\lambda\bar{\sigma} + \psi_3\left[\lambda\dot{\bar{\sigma}} + 2\dot{\lambda}\bar{\sigma}\right] = -\,\omega_1\left[\lambda^2 - \lambda^{-1}\right] - \omega_2\left[\lambda^4 - \lambda^{-2}\right]$$
$$-\,\frac{3}{2}\phi_1\left(\frac{\dot{\lambda}}{\lambda}\right) - \frac{3}{4}\phi_2\left(\frac{\dot{\lambda}}{\lambda}\right)^2 \,. \qquad (2.8)$$

At constant load, $\bar{\sigma}$ = constant, this is a first order differential equation for $\lambda(t)$ subject to an initial condition $\lambda(0) = 1 - e_e(\bar{\sigma})$. From Fig. 11 we see that $\lambda(t)$ is monotonic with $\dot{\lambda}$ a single-valued function of λ . This suggests we consider a simplified form of (2.8) with $\dot{\lambda}^2$ absent, by setting

$$\phi_2 \equiv 0 \,, \qquad (2.9)$$

though (2.9) is not a necessary condition for one real root $\dot{\lambda}$ on $0 < \lambda < 1$. At constant displacement rate, $\dot{\lambda}$ = constant, we have a first order differential equation for $\bar{\sigma}(t)$ with $\lambda = \dot{\lambda}t$ and $\bar{\sigma}(0) = 0$. The necessary $\dot{\bar{\sigma}}$ term to exhibit the time variation of Fig. 13 implies $\psi_3 \not\equiv 0$, though stress-rate dependence could instead be incorporated by dependence of other coefficients on $\dot{J}_2$, $\dot{J}_3$, with $\psi_3 \equiv 0$. Figure 14 shows the variation of $\dot{\bar{\sigma}}$ with e corresponding to the response in Fig. 13.

A small strain approximation neglecting $|e|$ compared to unity gives lead order expressions for uni-axial response

$$1 - \lambda \equiv e\ , -\dot{\lambda} \equiv \dot{e} = r,\ K_1 - 3 = 3e^2,\ K_2 - 3 = 3e^2, \qquad (2.10)$$

and $\psi_3, \phi_1, \omega_1, \omega_2$ can be regarded as functions of $\bar{\sigma}$, e if rate-dependence is restricted to the terms $\underset{\sim}{S}^{(1)}$ and $\underset{\sim}{D}$. The observed non-linear response at constant $\bar{\sigma}$ implies that the coefficients $\psi_3, \phi_1, \omega_1, \omega_2$ vary significantly with small changes of e, so cannot be approximated by their (constant) values at $K_1 = K_2 = 3$. To compare the magnitudes of various terms in (2.8) it is helpful to introduce a normalised strain and normalised time by

$$\bar{e} = Ce,\quad \bar{t} = t/t_o \qquad (2.11)$$

where C^{-1} is a maximum strain (·05 say) and $|\bar{\sigma}|$, $\left|\frac{d\bar{\sigma}}{d\bar{t}}\right|$ and $\left|\frac{d\bar{e}}{d\bar{t}}\right|$ are order unity in magnitude at their maxima. We suppose the stress unit is one bar and consider a range $|\bar{\sigma}| < 10$ bars. Now (2.8), (2.9) reduce to

$$\bar{\sigma} + \frac{\psi_3}{t_o}\frac{d\bar{\sigma}}{d\bar{t}} - C^{-1}\left[\bar{e}(3\omega_1 + 6\omega_2) + \frac{1}{t_o}\frac{d\bar{e}}{d\bar{t}}(2\psi_3\bar{\sigma} + \tfrac{3}{2}\phi_1)\right] = 0, \qquad (2.12)$$

and immediately $|C^{-1}\,\bar{\sigma}\,d\bar{e}/d\bar{t}| << d\bar{\sigma}/d\bar{t}$, but to retain the required dependence on $\underset{\sim}{S}^{(1)}$, $\underset{\sim}{D}$, and $\underset{\sim}{B}$, it is necessary to define order unity coefficients

$$\bar{\psi}_3 = \frac{\psi_3}{t_o}, \quad \bar{\phi}_1 = \frac{3\phi_1}{2Ct_o}, \quad \bar{\omega} = \frac{3\omega_1 + 6\omega_2}{C}, \qquad (2.13)$$

so that (2.12) becomes

$$\bar{\sigma} + \bar{\psi}_3\frac{d\bar{\sigma}}{d\bar{t}} - \bar{\omega}\bar{e} - \bar{\phi}_1\frac{d\bar{e}}{d\bar{t}} = 0\ . \qquad (2.14)$$

Uni-axial response cannot distinguish the ω_1, ω_2 terms, so we have three independent coefficients $\bar{\omega}(\bar{\sigma},\bar{e})$, $\bar{\phi}_1(\bar{\sigma},\bar{e})$, $\bar{\psi}_3(\bar{\sigma},\bar{e})$.

The differential relation (2.14) does not apply directly to a stress jump Δ_σ, but we can deduce the resulting strain jump Δ_e if we assume that (2.14) applies in the limit of a smooth rapid stress change from $\bar{\sigma}_o$ to $\bar{\sigma}_o + \Delta_\sigma$ and also that

$$\bar{e} = \bar{e}_o + h(\bar{\sigma}), \quad h(\bar{\sigma}_o) = 0, \qquad (2.15)$$

for any positive Δ_σ, accompanied by a strain change $\bar{e}_o$ to $\bar{e}_o + \Delta_e$ where $\Delta_e = h(\bar{\sigma}_o + \Delta_\sigma)$. Let the change take place in the time interval $(\bar{t}_o, t_o + \delta t)$, integrate (2.14) over the time interval, and let $\delta t \to 0$. Since $\bar{\omega}$ is bounded we obtain

$$\int_{\bar{\sigma}_o}^{\bar{\sigma}_o+\Delta_\sigma} \bar{\psi}_3\left[\bar{\sigma},\bar{e}_o + h(\bar{\sigma})\right]d\bar{\sigma} = \int_{\bar{\sigma}_o}^{\bar{\sigma}_o+\Delta_\sigma} h'(\bar{\sigma})\bar{\phi}_1\left[\bar{\sigma},\bar{e}_o + h(\bar{\sigma})\right]d\bar{\sigma}, \qquad (2.16)$$

for all Δ_σ, and hence

$$\bar{\phi}_1[\bar{\sigma},\bar{e}_o + h(\bar{\sigma})]h'(\bar{\sigma}) - \bar{\psi}_3[\bar{\sigma},\bar{e}_o + h(\bar{\sigma})] = 0, \quad h(\bar{\sigma}_o) = 0. \tag{2.17}$$

Thus, given $\bar{\phi}_1$, $\bar{\psi}_3$, (2.17) is a first order differential equation with initial condition for $h(\bar{\sigma})$, and the solution determines $\Delta_e = h(\bar{\sigma}_o + \Delta_\sigma)$ for all Δ_σ. The strain jump depends on the stress jump and the starting stress. In particular, if the jump Δ is applied at time $t = 0$ from zero stress and strain, Δ_e is given by setting $\bar{e}_o = \bar{\sigma}_o = 0$ in (2.17), and determines the elastic strain $e_e(\Delta) = C^{-1}\Delta_e(\Delta)$. However, the uni-axial response for all Δ determines $\Delta_e(\Delta) = h(\Delta)$, and so in principle determines a relation between $\bar{\phi}_1$ and $\bar{\psi}_3$ in the strain range $\bar{e} = 0 \rightarrow C\,e_e$ not covered by the smooth response.

<u>Constant load and constant displacement rate</u>.

For constant $\bar{\sigma}$ in $t > 0$, (2.14) becomes

$$\bar{\phi}_1 \frac{d\bar{e}}{d\bar{t}} + \bar{\omega}\,\bar{e} - \bar{\sigma} = 0 . \tag{2.18}$$

Given a family of normalised curves corresponding to Fig. 12,

$$\frac{d\bar{e}}{d\bar{t}} = F(\bar{\sigma},\bar{e}) , \tag{2.19}$$

then

$$\bar{\sigma} - \bar{\omega}\,\bar{e} = \bar{\phi}_1 F , \quad \bar{e} \geqslant \bar{e}_e(\bar{\sigma}) \tag{2.20}$$

determines one relation between $\bar{\omega}$ and $\bar{\phi}_1$. For constant displacement rate defined by $d\bar{e}/d\bar{t} = w$, so $\bar{e} = w\bar{t}$, (2.14) becomes

$$\bar{\psi}_3 \frac{d\bar{\sigma}}{d\bar{t}} + \bar{\sigma} - \bar{\omega}\bar{e} - \bar{\phi}_1 w = 0 . \tag{2.21}$$

Given families of normalised curves corresponding to Figs 13 and 14,

$$\bar{\sigma} = g(w,\bar{e}), \quad \frac{d\bar{\sigma}}{d\bar{t}} = w\frac{\partial g}{\partial \bar{e}} = G(w,\bar{e}), \quad \bar{e} \geqslant 0 , \tag{2.22}$$

then

$$\bar{\psi}_3\left[g(w,\bar{e}),\bar{e}\right]G(w,\bar{e}) = \bar{\phi}_1\left[g(w,\bar{e}),\bar{e}\right]\{-F\left[g(w,\bar{e}),\bar{e}\right] + w\} \tag{2.23}$$

determines $\bar{\psi}_3$ *in terms of* $\bar{\phi}_1$. *Since* $F > 0$, $w > 0$, and $G = 0$ at $\bar{e} = \bar{e}_M(w)$, and assuming $\bar{\phi}_1(\bar{\sigma},\bar{e})$ has no zero, then

$$F\left[g\{w,\bar{e}_M(w)\}, \bar{e}_M(w)\right] = w . \tag{2.24}$$

Now (2.20), (2.23) provide two relations for three functions $\bar{\psi}_3$, $\bar{\omega}$, $\bar{\phi}_1$, all of which are required to describe the responses in Figs 11 and 13 and incorporate dependence on the reference configuration through $\underset{\sim}{B}$. Clearly $\omega \equiv 0$, $\bar{\sigma} = \bar{\phi}_1 F$, solve (2.20), but induced anisotropy is no longer possible. In this case, however, $\bar{\phi}_1$ and $\bar{\psi}_3$ are determined by F and G, and, alternatively, complete responses F and G can in principle be matched by suitable choice of $\bar{\phi}_1$ and $\bar{\psi}_3$. When $\bar{\omega} \neq 0$ there is additional flexibility.

While some values of $g = \bar{\sigma}$ are repeated on the two sides of $\bar{e} = \bar{e}_M(w)$, illustrated in Fig. 13, the arguments $(g,\bar{e})$ corresponding to the same g on a constant w curve are distinct points of the $(\bar{\sigma},\bar{e})$ domain. Further, assuming that $\partial g/\partial w|_{\bar{e}} > 0$, the family of curves are non-intersecting, so each constant w relation (2.23) between $\bar{\psi}_3$ and $\bar{\phi}_1$ refers to distinct sets of points $(\bar{\sigma},\bar{e})$. This situation contrasts strongly with the minimal fluid model in section 1 which was not sufficiently general to match independent constant stress and constant strain-rate data.

A restricted model $\bar{\phi}_1 = \bar{\phi}_1(\bar{e})$, $\bar{\omega} = \bar{\omega}(\bar{e})$ satisfies (2.20) provided that

$$F = \bar{\sigma} f_1(\bar{e}) + f_o(\bar{e}), \quad \bar{\phi}_1 = 1/f_1, \quad \bar{\omega} = -\bar{\phi}_1 f_o/\bar{e} , \tag{2.25}$$

which requires the strain-rate at fixed strain to change <u>linearly</u> with the constant stress $\bar{\sigma}$. A positive $f_1(\bar{e})$ would imply a linearly increasing strain-rate which we expect to be qualitatively correct. With the restrictions (2.25), (2.23) becomes

$$\bar{\psi}_3\left[g(w,\bar{e}),\bar{e}\right]G(w,\bar{e})f_1(\bar{e}) = w - f_o(\bar{e}) - f_1(\bar{e})g(w,\bar{e}), \tag{2.26}$$

which requires that $\bar{\psi}_3$ depends on both $\bar{\sigma}$ and $\bar{e}$ in general.

Writing $\bar{\phi}_1 = E\bar{\psi}_3$, (2.23) becomes

$$\frac{\partial g(w,\bar{e})}{\partial \bar{e}} = E\left[g(w,\bar{e}),\bar{e}\right]\{1 - \frac{F\left[g(w,\bar{e}),\bar{e}\right]}{w}\} \tag{2.27}$$

which interprets $E(\bar{\sigma},\bar{e})$ as a Young's modulus in the limit of inifinite strain-rate. A restriction E = constant implies

a relation between the supposed independent functions $F(\bar{\sigma},\bar{e})$, $g(w,\bar{e})$; that is from (2.22), (2.27),

$$\left.\frac{\partial G}{\partial \bar{e}}\right|_{w} = -E\left\{\frac{G}{w}\left.\frac{\partial F}{\partial \bar{\sigma}}\right|_{\bar{e}} + \left.\frac{\partial F}{\partial \bar{e}}\right|_{\bar{\sigma}}\right\} . \tag{2.28}$$

If $\bar{e} = \bar{e}_1(\bar{\sigma})$ corresponds to the second inflexion point in Fig. 11, or local maximum $\dot{e}$ in Fig. 12, and $\bar{e} = \bar{e}_2(w)$ corresponds to the inflexion point in Fig. 13, or local minimum $\dot{\bar{\sigma}}$ in Fig. 14, then

$$\left.\frac{\partial F}{\partial \bar{e}}\right|_{\bar{\sigma}} \begin{cases} < 0 & 0 < \bar{e} < \bar{e}_m \\ > 0 & \bar{e}_m < \bar{e} < \bar{e}_1 \\ < 0 & \bar{e}_1 < \bar{e} , \end{cases} \qquad \left.\frac{\partial G}{\partial \bar{e}}\right|_{w} \begin{cases} < 0 & 0 < \bar{e} < \bar{e}_2 \\ > 0 & \bar{e}_2 < \bar{e} . \end{cases} \tag{2.29}$$

Adopting the anticipated inequalities $\partial F/\partial\bar{\sigma}|_{\bar{e}} > 0$ and $e_2 > e_m$, then the inequalities (2.29) are certainly consistent with (2.28) for $e_m \leqslant \bar{e} \leqslant \min\ (\bar{e}_1,\bar{e}_2)$, but no firm conclusion can be drawn in the other ranges of $\bar{e}$. Thus E = constant may, or may not, be an acceptable model. Given independent F,g, (2.27) determines $E(\bar{\sigma},\bar{e})$, and (2.20) relates $\bar{\omega}$ to $\bar{\phi}_1$ or $\bar{\psi}_3$.

The next stage of our investigation must explore various simplications to find $\bar{\phi}_1$, $\bar{\psi}_3$, $\bar{\omega}$, satisfying (2.20), (2.23), for F,g,G compatible with expected qualitative response. Once such a model is determined we wish to demonstrate the anisotropy induced by simple loading histories.

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Figure Captions

1. Strain-rate $r(t)$ at constant stress σ .
2. Stress $\sigma(t)$ at constant strain-rate r.
3. Dependence domains of response functions.
4. Functions $\overline{f}(\alpha_1)$, $\overline{F}(\alpha_2)$, $1/q(\sigma)$, $1/s_1(r)$, $1/s_2(r)$ for family 1.
5. Strain-rates $r(t)$ at constant $\sigma = \sigma_j = 2(-0{\cdot}25)0{\cdot}5$ bars, for family 1 (———), predicted by reduced model (-----). r decreases at fixed t as σ_j decreases.
6. Strains $\varepsilon(t)$ corresponding to strain-rates shown in Fig. 5.
7. Stress-strain curves at constant $r = r_m(\sigma_j)$ predicted by reduced model for family 1 with $s = s_1$ (———) and $s = s_2$ (-----).
8. Strain-rates $r(t)$ at constant $\sigma = \sigma_j = 2(-0{\cdot}25)0{\cdot}75$ bars, for family 2 (———), predicted by reduced model with $q = q_1$ (-----) and with $q = q_2$ (.....).
9. Strains $\varepsilon(t)$ corresponding to strain-rates shown in Fig. 8.
10. Stress-strain curves at constant $r = r_m(\sigma_j)$ predicted by reduced model for family 2 with $q = q_1$ and $s = s_1$(————), $s = s_2$ (-----), $s = s_3$(......).
11. Compressive strain response at constant load.
12. Strain-rate v. strain at constant load.
13. Stress-strain curve at constant displacement-rate.
14. Stress-rate v. strain at constant displacement-rate.

Fig. 1. Strain-rate r(t) at constant stress σ .

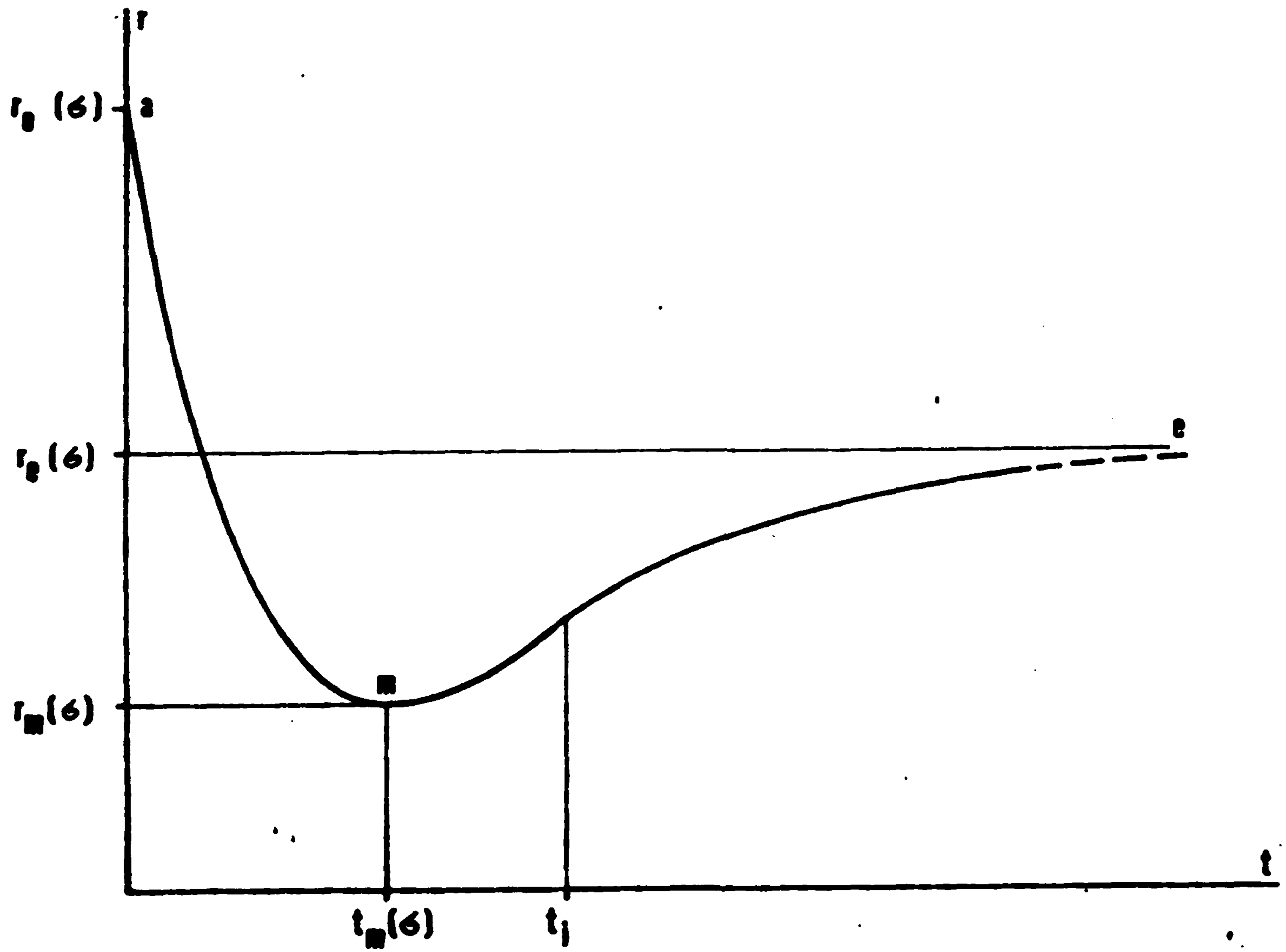
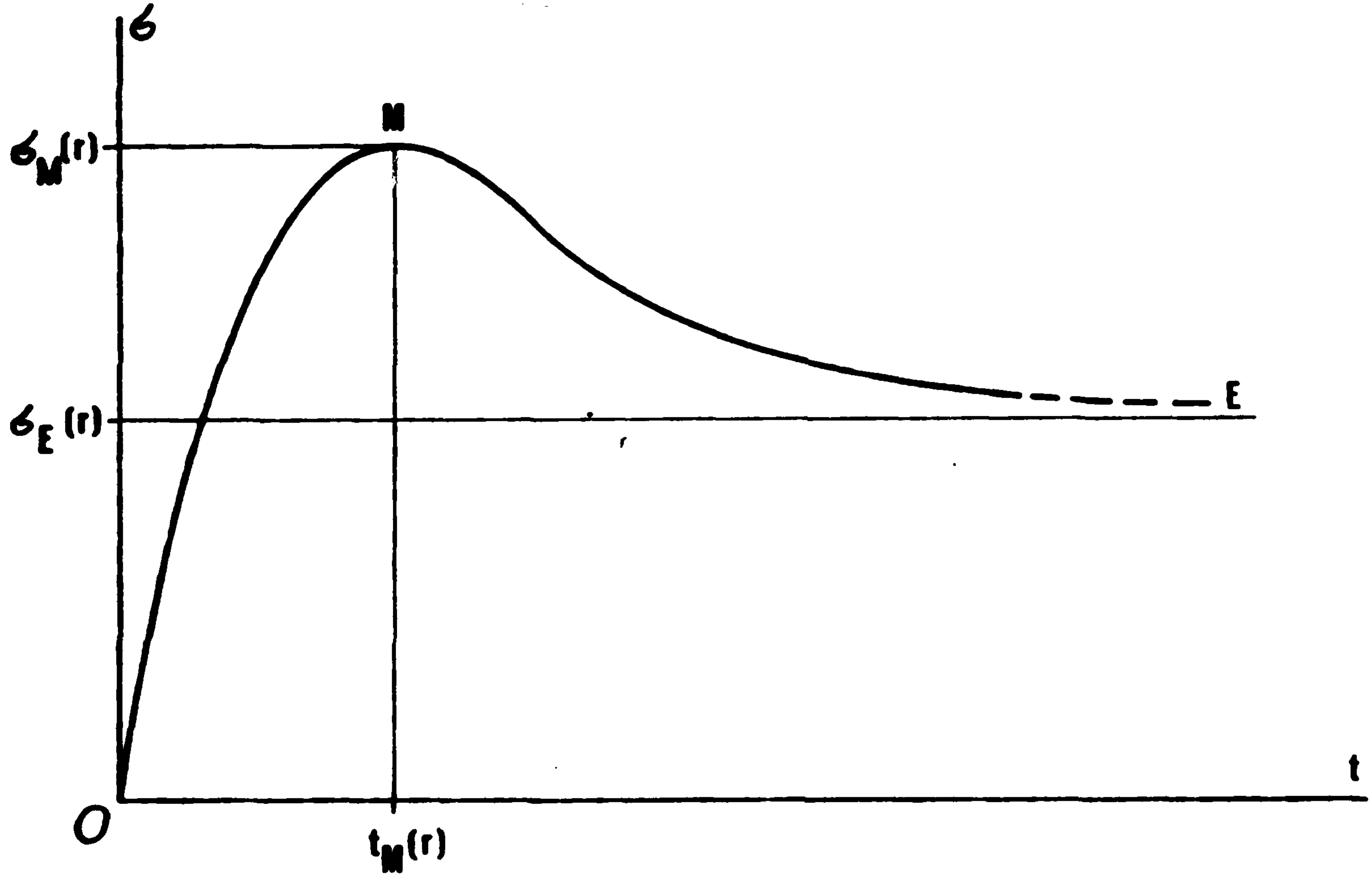


Fig. 2. Stress $\sigma(t)$ at constant strain-rate r.



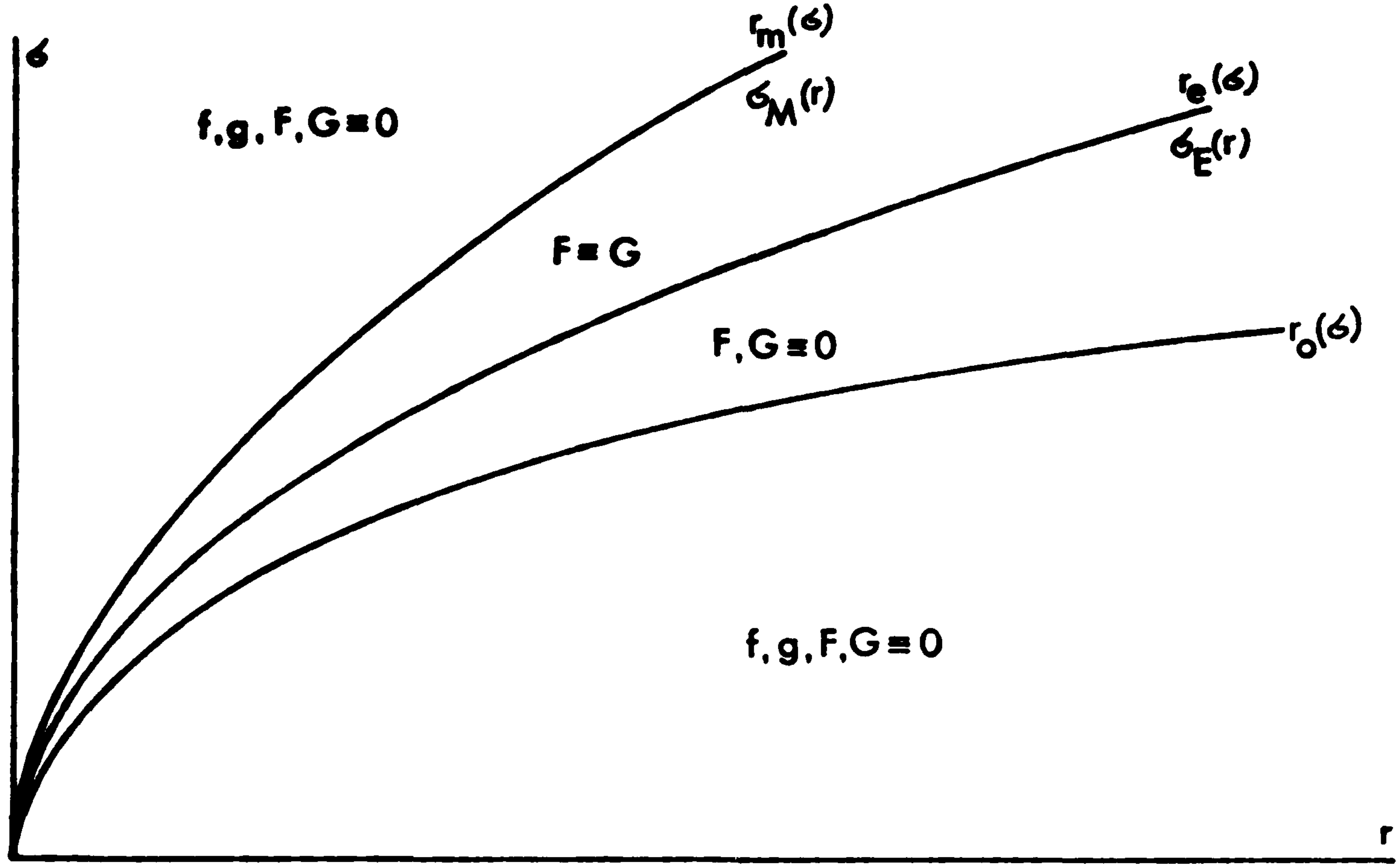


Fig. 3. Dependence domains of response functions.

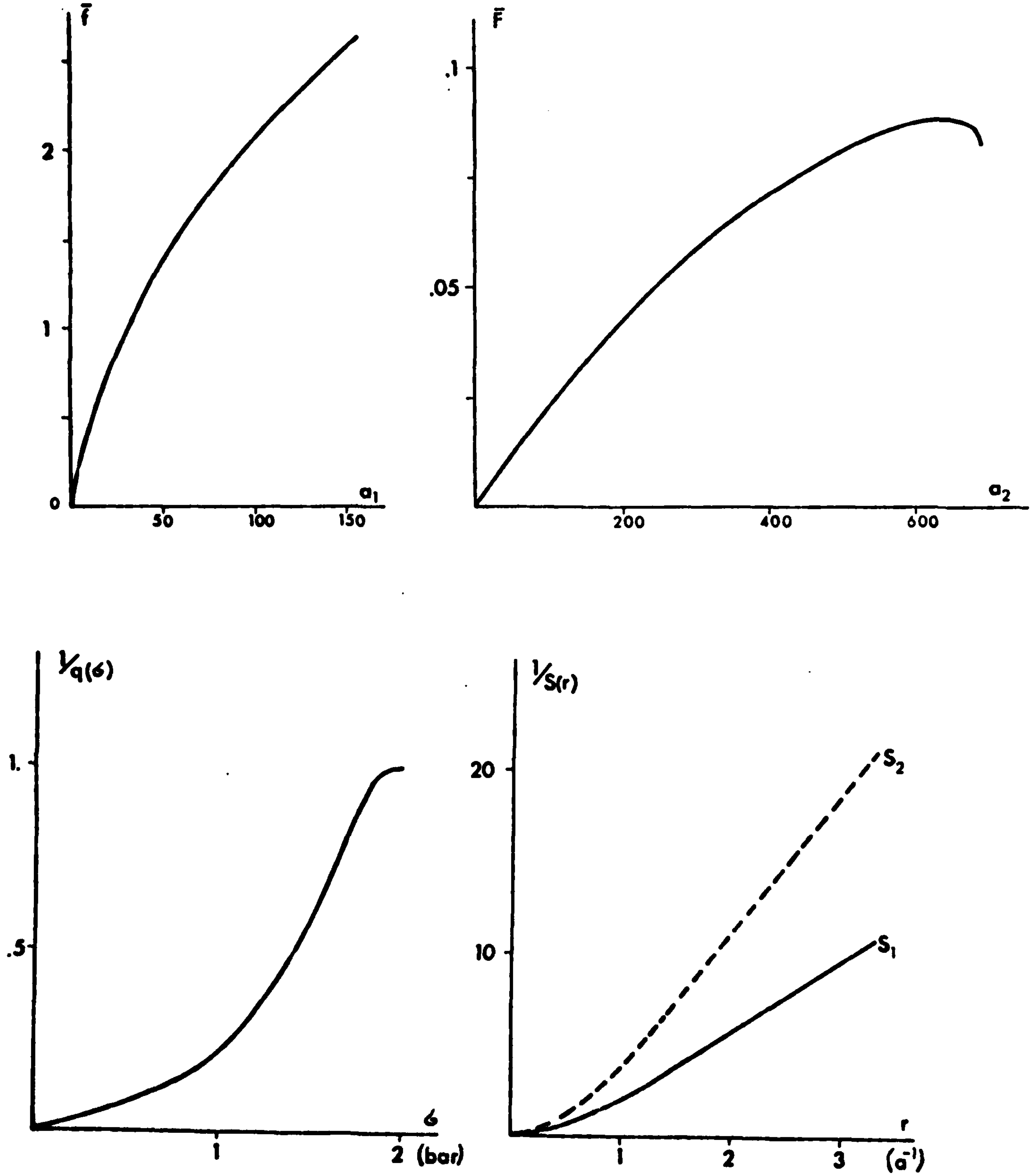


Fig. 4. Functions $\bar{f}(a_1)$, $\bar{F}(a_2)$, $1/q(\sigma)$, $1/s_1(r)$, $1/s_2(r)$ for family 1.

Fig. 5. Strain-rates $r(t)$ at constant $\sigma = \sigma_j = 2(-0{\cdot}25)0{\cdot}5$ bars, for family 1 (———), predicted by reduced model (-----). r decreases at fixed t as σ_j decreases.

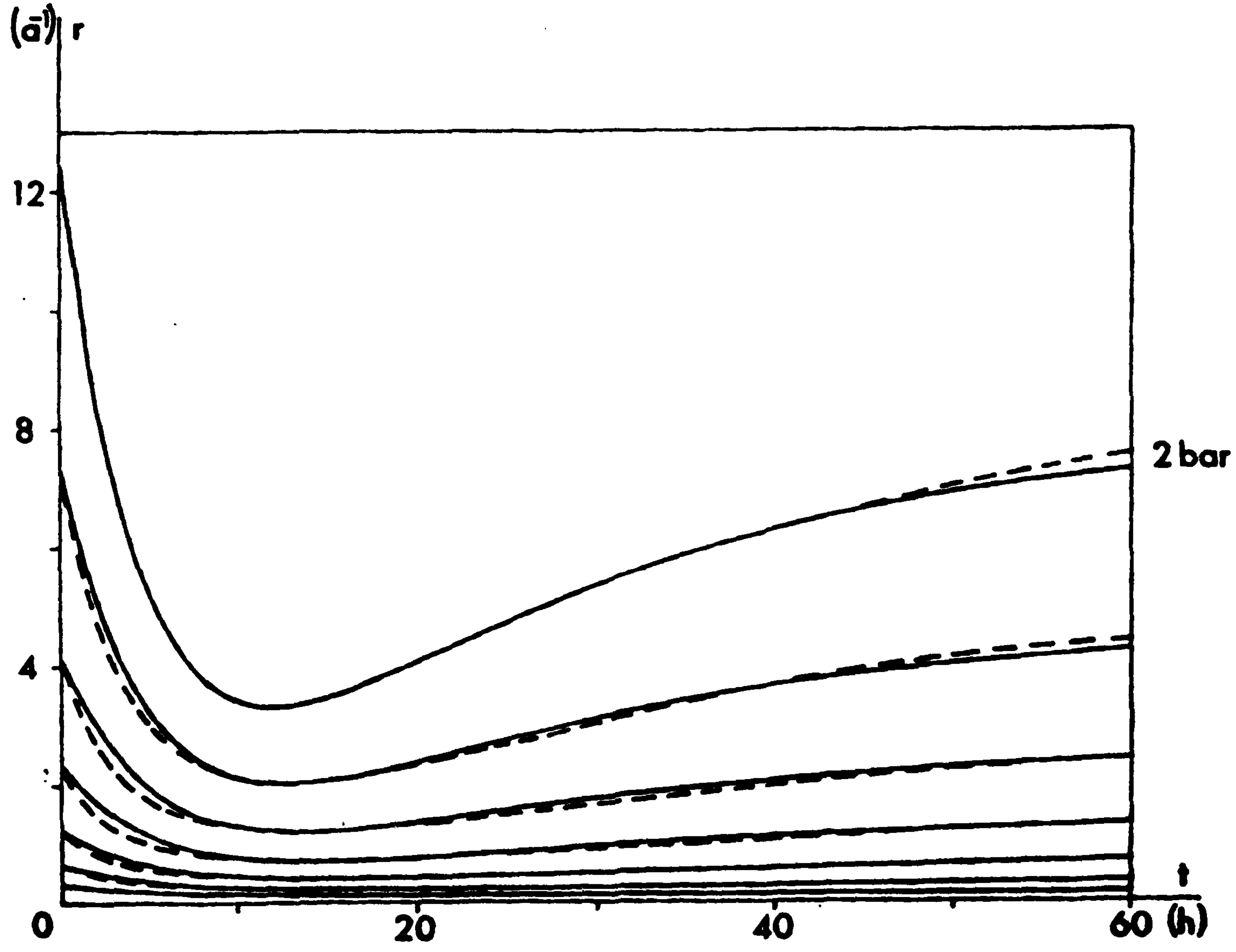
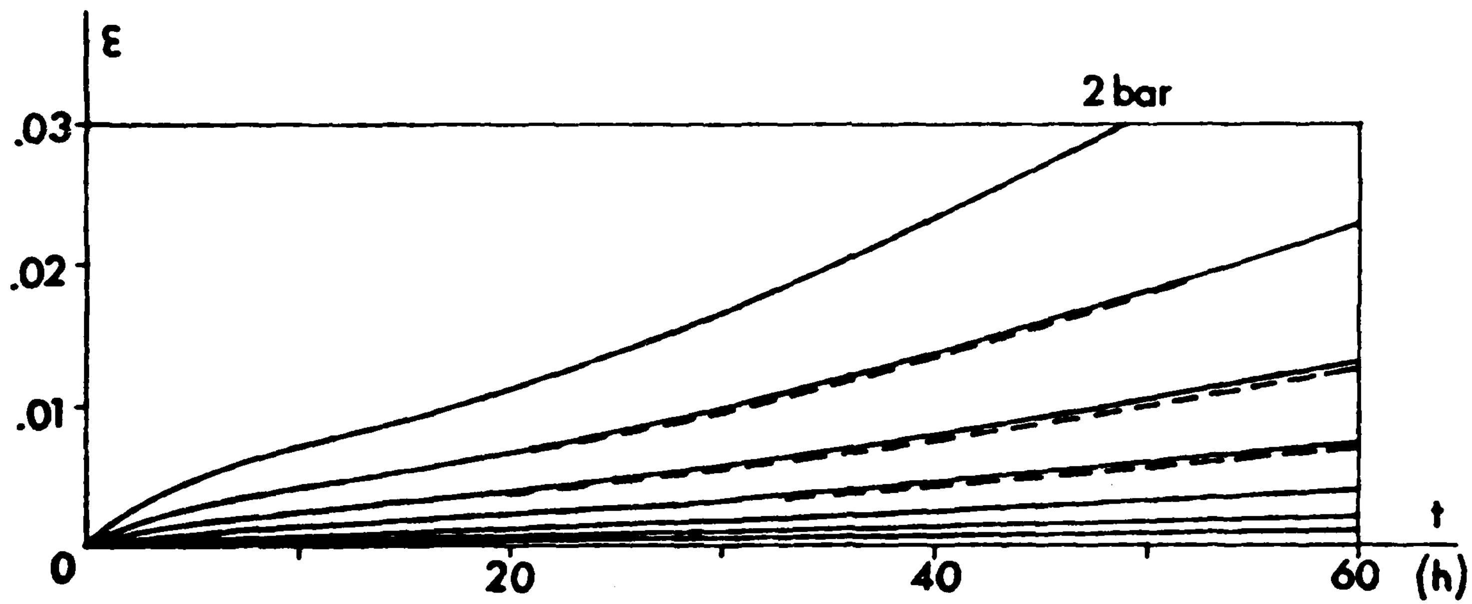


Fig. 6. Strains $\varepsilon(t)$ corresponding to strain-rates shown in Fig. 5.



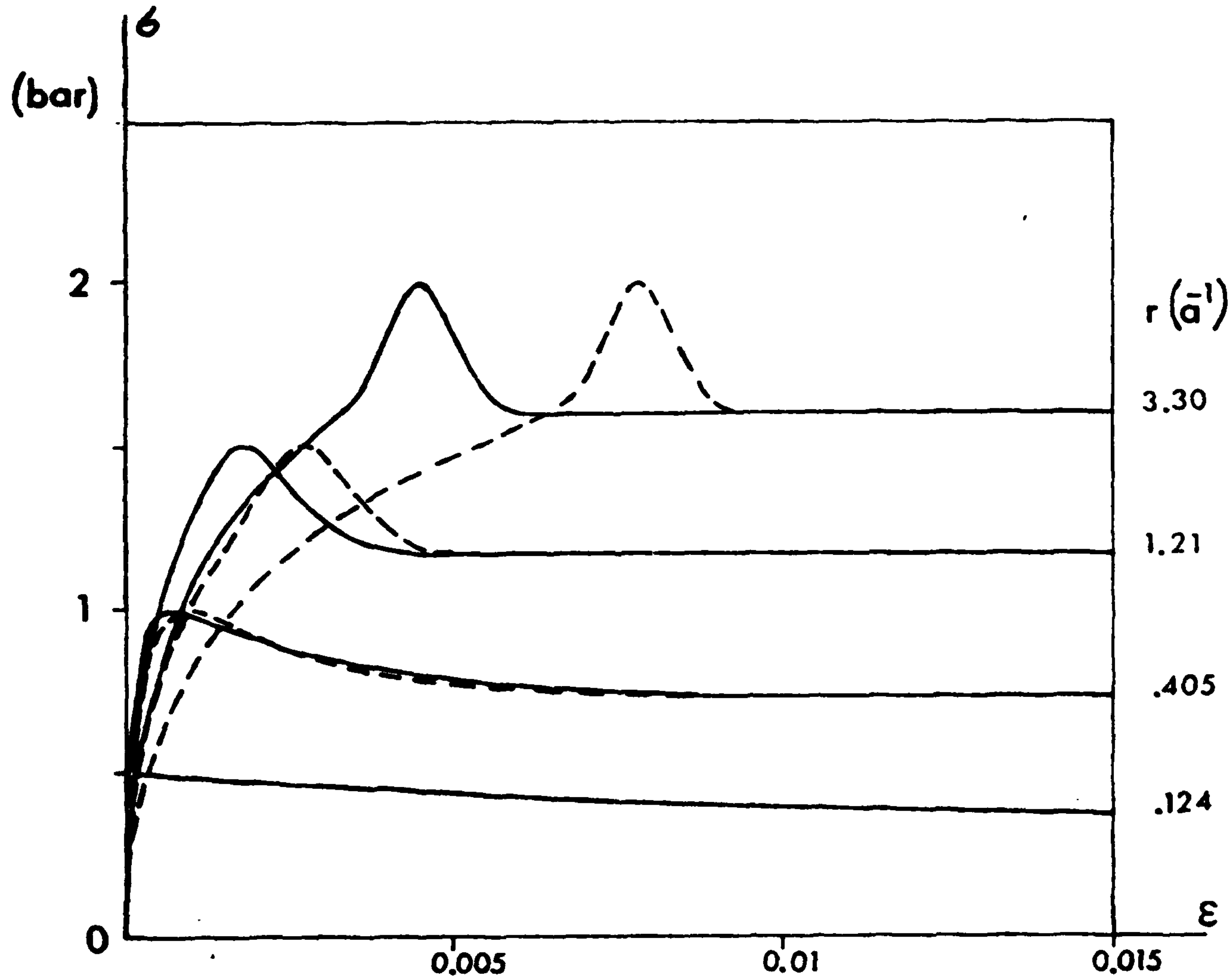


Fig. 7. Stress-strain curves at constant $r = r_m(\sigma_j)$ predicted by reduced model for family 1 with $s = s_1$ (———) and $s = s_2$(-----).

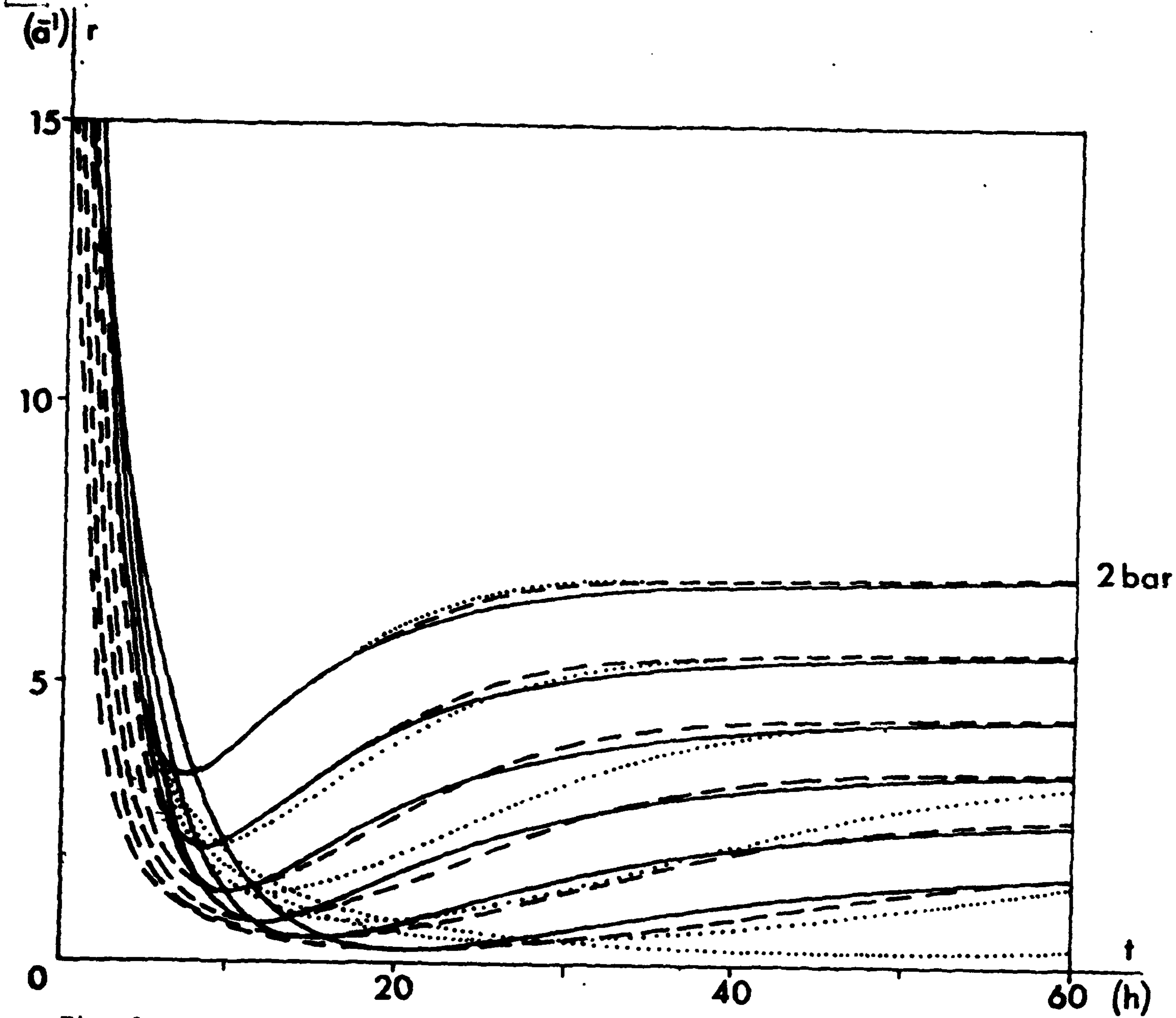


Fig. 8. Strain-rates $r(t)$ at constant $\sigma = \sigma_j = 2(-0{\cdot}25)0{\cdot}75$ bars, for family 2 (———), predicted by reduced model with $q = q_1$(-----) and with $\sigma = q_2$ (.....).

Fig. 9. Strains $\varepsilon(t)$ corresponding to strain-rates shown in Fig. 8.

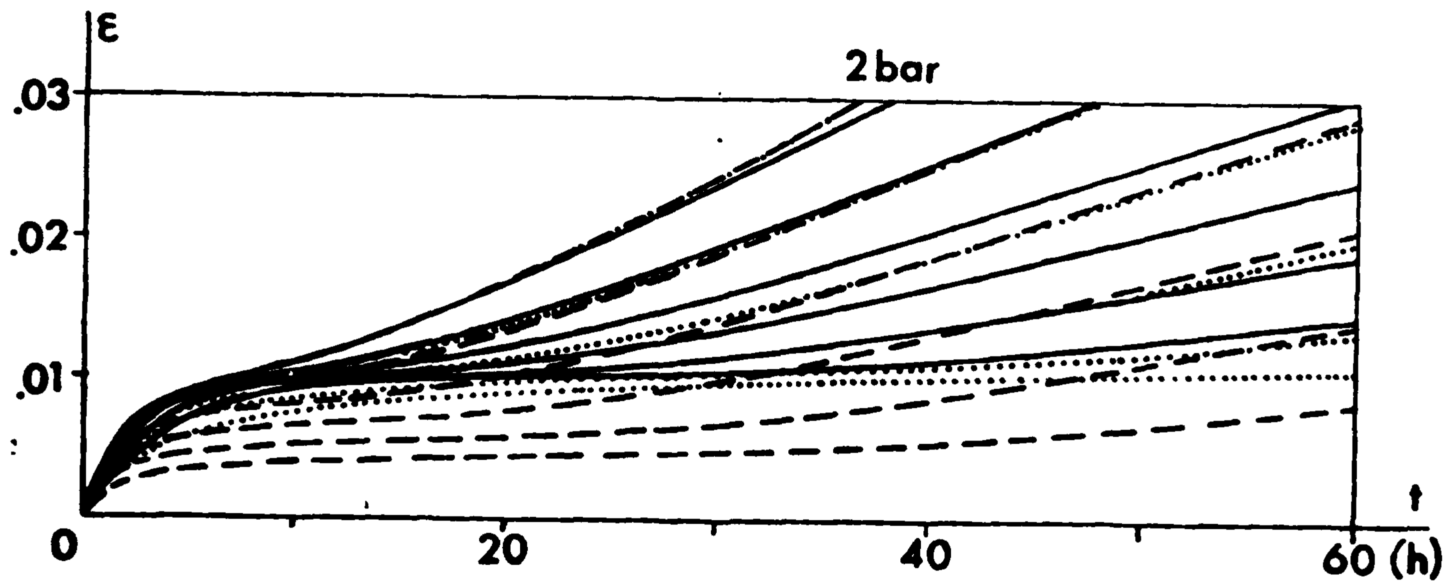


Fig. 10. Stress-strain curves at constant $r = r_m(\sigma_j)$ predicted by reduced model for family 2 with $q = q_1$ and $s = s_1$ (———), $s = s_2$ (-----), $s = s_3$ (.....).

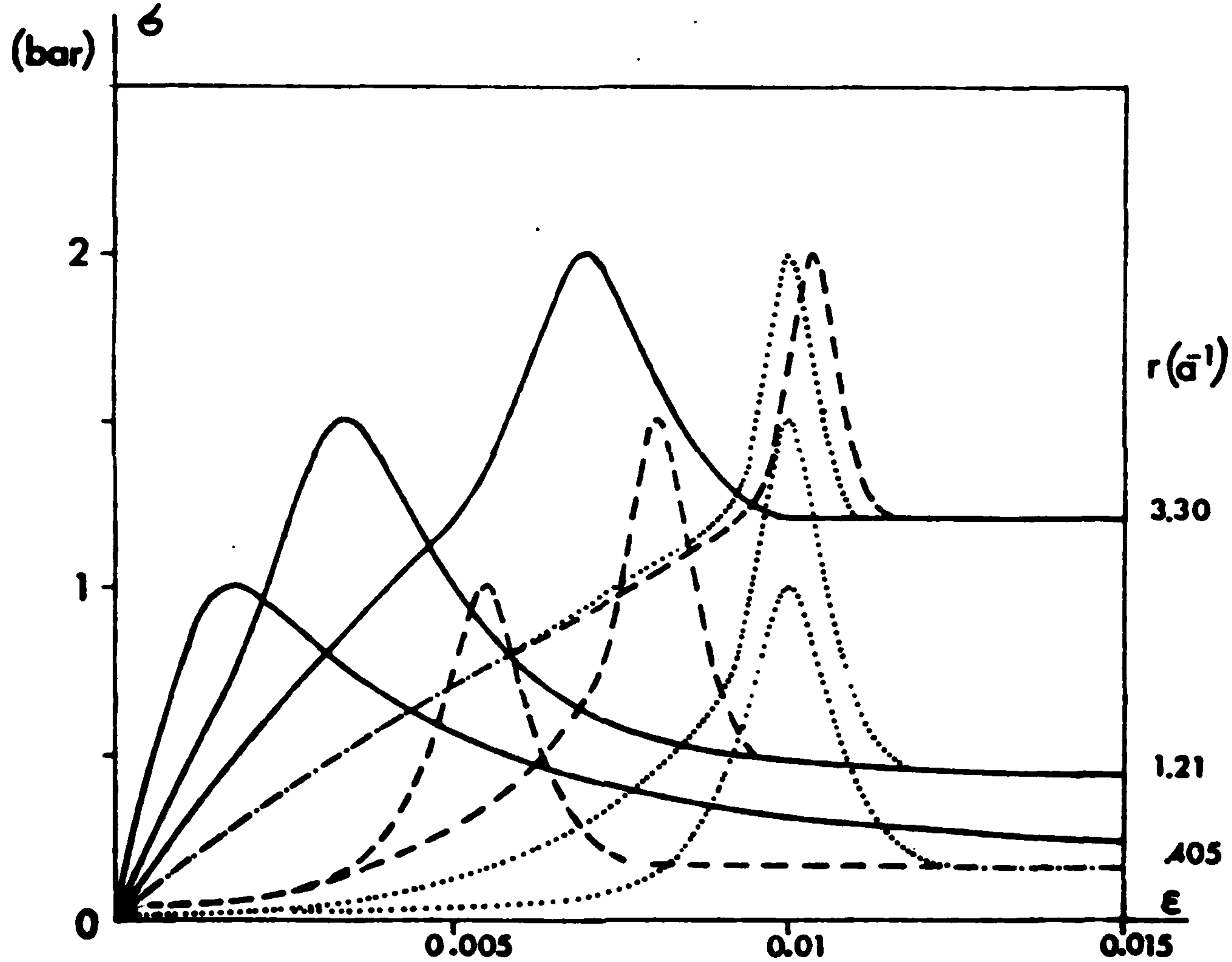


Fig. 11. Compressive strain response at constant load.

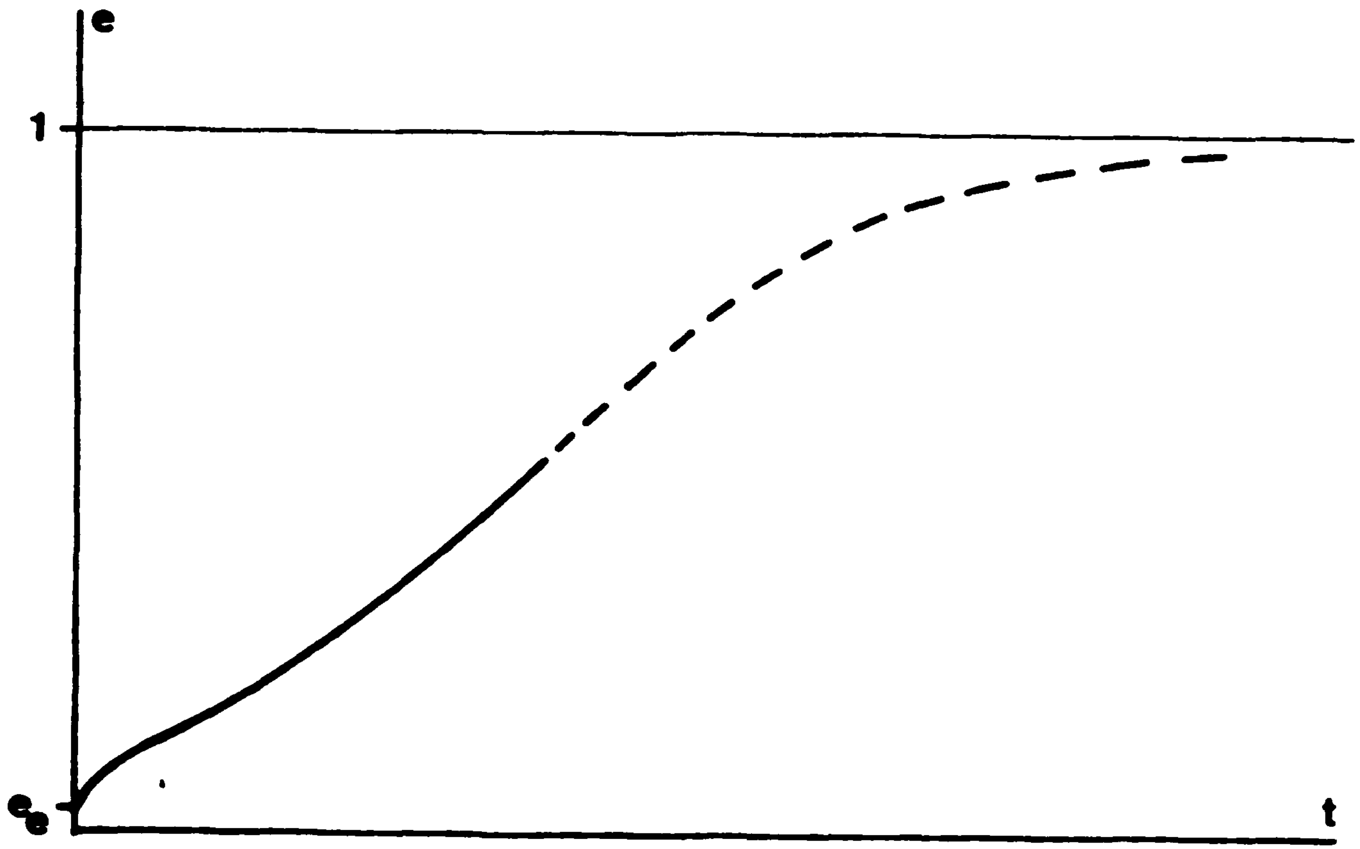


Fig. 12. Strain-rate v. strain at constant load.

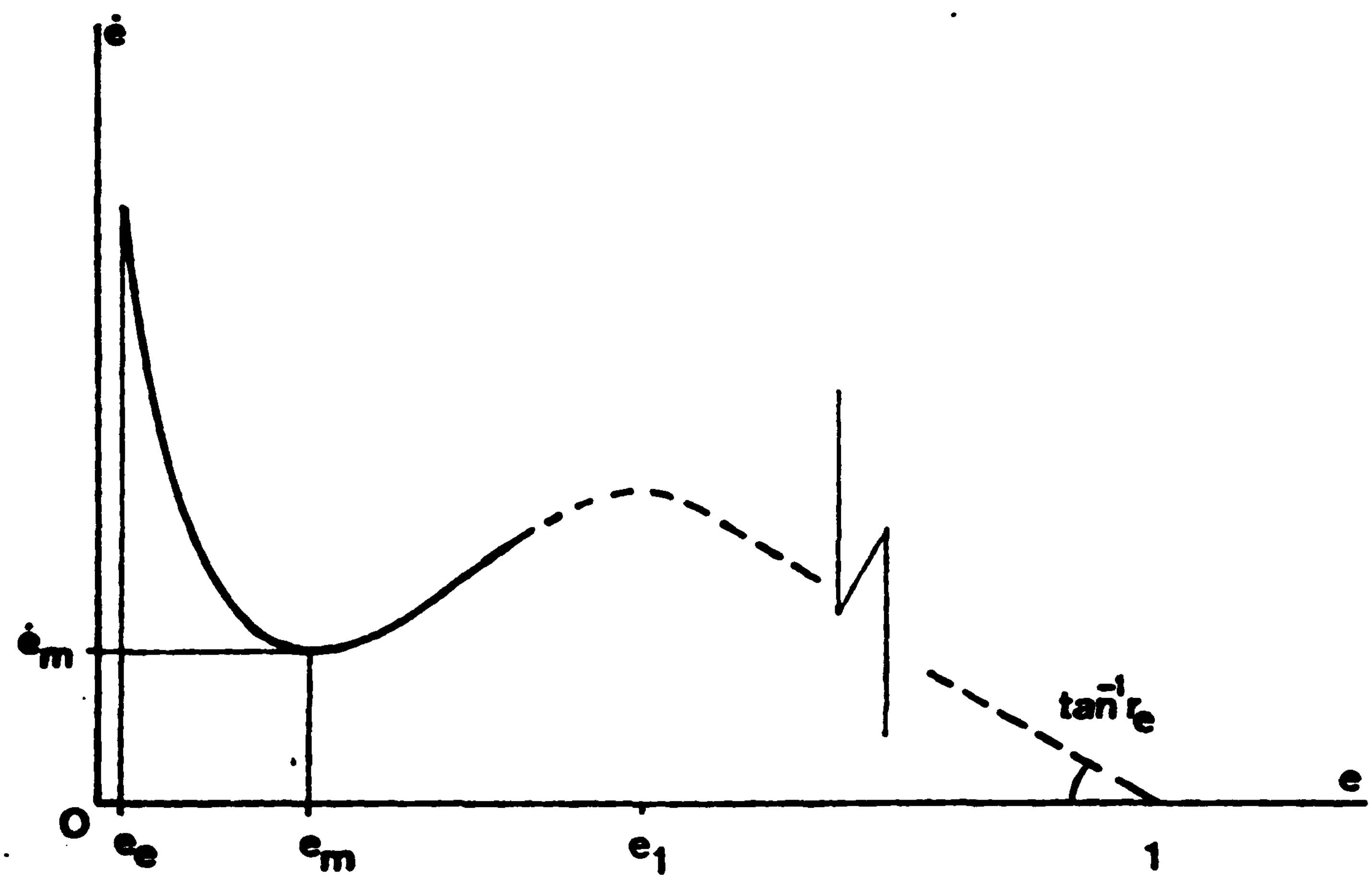


Fig. 13. Stress-strain curve at constant displacement-rate.

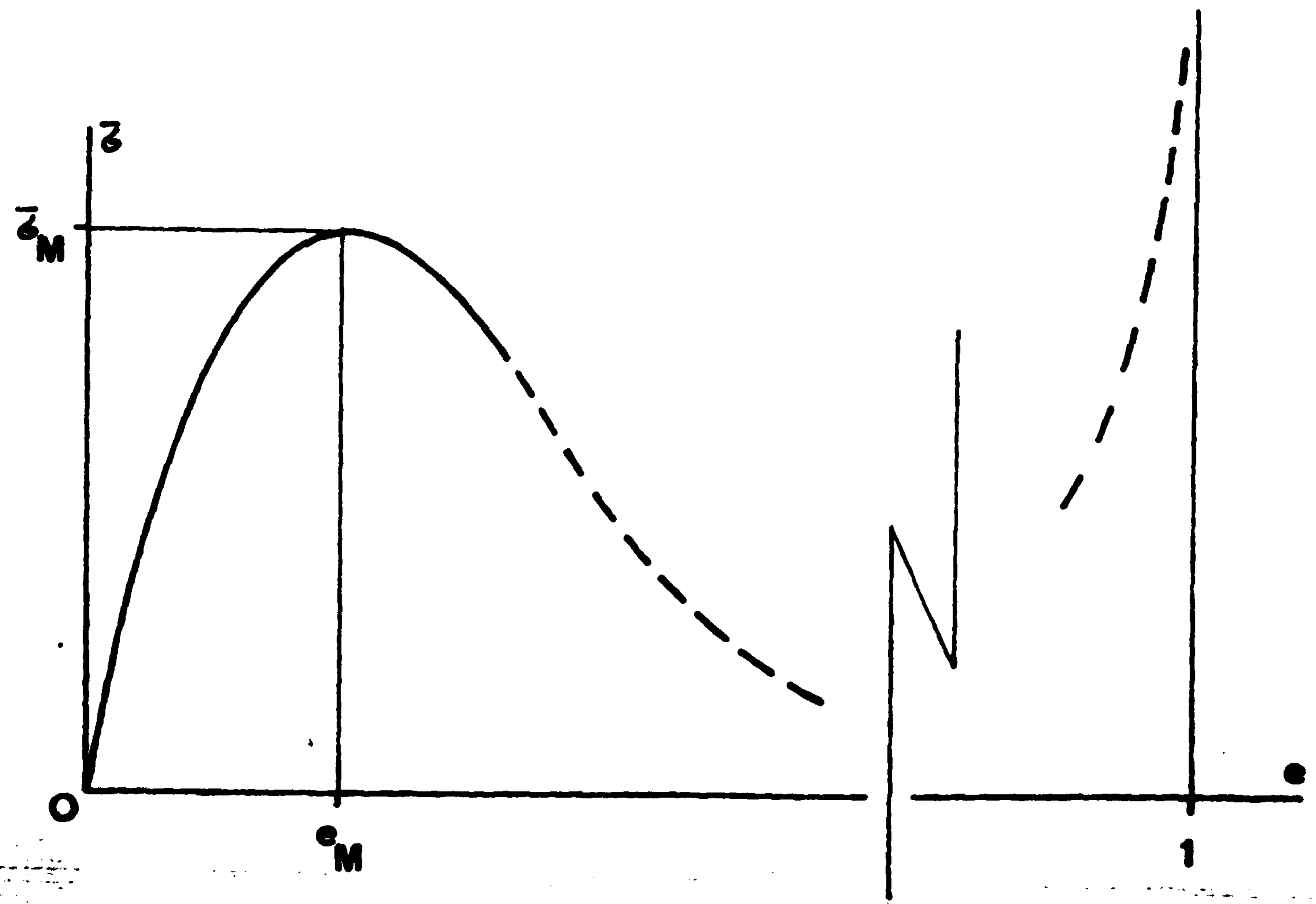


Fig. 14. Stress-rate v. strain at constant displacement-rate.